2

Introduction to the Principles of

RELATIVISTIC FIELD THEORY

Introduction. We have several times taken passing notice of what appeared to be a natural "relativistic predisposition" of classical field theory—of field theories in general. The reason for this state of affairs is not far to find: it was clearly articulated more than ninety years ago by Hermann Minkowski, who in 1908 had occasion to speak as follows:[1]

> *"The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."*

In the dynamics of particles, as formulated by Newton and carried to a kind of perfection by Lagrange/Hamilton, the space/time distinction remains vivid; one is concerned in that theory with expressions $\boldsymbol{x}(t)$ into which $\boldsymbol{x}$ enters as the dependent variable, t (recorded by "the clock on God's wall") as the independent variable. In field theories, on the other hand, one is concerned with expressions of the form $\varphi(\boldsymbol{x}, t)$; the spatial variables $\boldsymbol{x}$ have joined the temporal variable t in an *expanded list of conceptually distinct but formally co-equal independent variables*. "Spacetime" has come into being as the 4-dimensional plenum upon which field theories are written.

---

[1] I quote from the introduction to his "Space and time," which is the text of an address delivered on 21 September 1908 before the 80^{th} Assembly of German Natural Scientists and Physicians, in Cologne. The paper, in English translation, can be found *The Principle of Relativity* (1923), which is available as a Dover reprint.

This is not to say that field theory is "automatically relativistic." It remains to turn the "plenum" into a specifically structured metric manifold, as also it remains to stipulate that inertial observers are interconnected by transformations which preserve that metric structure. But field theory is predisposed to favor such developments. . . and historically it was a field theory (electrodynamics) which stimulated those developments—developments which led to articulation of the Principle of Relativity.

To phrase the point another way: Hamilton's principle, as encountered in particle mechanics

$$\delta\int_{\text{time interval}} L\,dt = 0$$

is anti-relativistic in that it assigns a preferred place to the temporal variable t. But its field-theoretic counterpart

$$\delta\iiiint_{\text{spacetime bubble}} \mathcal{L}\,dxdydzdt = 0$$

is "pro-relativistic" in the obvious sense that it assigns formally identical roles to each of the spacetime coordinates.

So diverse are the distributed systems encountered in Nature that field theory has an unruly tendency to sprawl. Even after discarding all aspects of the topic (some physically quite important) which fall outside the ruberic of "Lagrangian field theory," one is left with potential subject matter far too vast to be surveyed in thirty-six lectures.[2] On a previous occasion, seeking to further condense the subject, I chose to treat only relativistic fields,[3] but found even that restricted topic to be much too broad (and in many respects too advanced) for comprehensive treatment in such a setting. Here I propose to examine only some introductory aspects of relativistic classical field theory. The specific systems we will be discussing have been selected to expose characteristic points of principle and methodology, and to provide the basic stock of concrete examples upon which we will draw in later work.

Notational conventions & relativistic preliminaries. Honoring an almost universal convention, we will use Greek indices to distinguish spacetime coordinates x^μ. Specifically

$$\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}$$

---

[2] On can, in this light, understand why L. D. Landau & E. M. Lifshitz, writing under the title *The Classical Theory of Fields* (2nd edition 1962), elect actually to treat only two fields: the electromagnetic field and the gravitational field.

[3] RELATIVISTIC CLASSICAL FIELDS (1973).

Spacetime acquires its Minkowskian metric structure from the metric tensor $g_{\mu\nu}$, which is taken to be given (in all frames) by[4]

$$\boldsymbol{g} \equiv \|g_{\mu\nu}\| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

Elements of the matrix $\boldsymbol{g}^{-1}$ inverse to $\boldsymbol{g}$ are denoted $g^{\mu\nu}$; thus $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu$ where the Einstein summation convention $\sum_\alpha$ is (as always) understood. We use $g^{\mu\nu}$ and $g_{\mu\nu}$ to raise and lower indices: thus $x_\mu = g_{\mu\alpha}x^\alpha$, etc. The Lorentzian inner product of a pair of 4-vectors x and y is defined/denoted

$$(x, y) \equiv x^{\mathsf{T}}\boldsymbol{g}\, y = x^\alpha g_{\alpha\beta} y^\beta = x^\alpha y_\alpha = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$$

Special relativity contemplates the response of physical theory to *linear* transformations $x^\mu \longrightarrow X^\mu = \Lambda^\mu{}_\alpha x^\alpha$ which *preserve the postulated metric structure of spacetime*. Linearity entails that the elements of the transformation matrix $M^\mu{}_\nu \equiv \partial X^\mu/\partial x_\nu = \Lambda^\mu{}_\nu$ are x-independent *constants*, and therefore that contravariant vectors transform *like coordinates*:

$$\begin{aligned} V^\mu &= \tfrac{\partial X^\mu}{\partial x^\alpha} v^\alpha \\ &= \Lambda^\mu{}_\alpha v^\alpha \end{aligned} \tag{2}$$

This circumstance introduces an enormous formal simplification into relativity. "Preservation of metric structure" is interpreted to mean that

$$g_{\mu\nu} \longrightarrow G_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu g_{\alpha\beta} \quad \text{gives } g_{\mu\nu} \text{ back again:}$$

$$\boldsymbol{\Lambda}^{\mathsf{T}}\boldsymbol{g}\,\boldsymbol{\Lambda} = \boldsymbol{g} \tag{3}$$

Every inertial observer, when asked to write out the metric tensor, writes the same thing; $g_{\mu\nu}$ has become a *universally available shared commodity* (like the Kronecker tensor $\delta^\mu{}_\nu$, and like the Levi-Civita tensor density $\epsilon_{\kappa\lambda\mu\nu}$). From this central fact (which serves to distinguish the *Lorentz transformations* from all other linear transformations) follows the Lorentz-invariance of all inner products:

$$\begin{aligned} (x, y) &= \boldsymbol{x}^{\mathsf{T}}\boldsymbol{g}\,\boldsymbol{y} \\ &= \boldsymbol{x}^{\mathsf{T}}\boldsymbol{\Lambda}^{\mathsf{T}}\boldsymbol{g}\,\boldsymbol{\Lambda}\,\boldsymbol{y} \\ &= \boldsymbol{X}^{\mathsf{T}}\boldsymbol{g}\,\boldsymbol{Y} \equiv (\boldsymbol{X}, \boldsymbol{Y}) \end{aligned} \tag{4}$$

[4] It has been my practice for more than forty years to use doublestroke characters to distinguish matrices from other kinds of mathematical objects; thus $\mathbb{M} = \|M_{ij}\|$. But TEX provides a very limited set of such characters. Forced to abandon my former convention, I will here use **boldface** to accomplish that distinction.

In the case $y = x$ we obtain the Lorentz invariance of

$$s^2 \equiv (x, x) \quad : \quad \text{the Lorentzian "squared length" of } x$$

We are by this point in touch with the hyperbolic geometry of spacetime—in touch, that is to say, with the standard stuff of textbook relativity (time dilation, length contraction, breakdown of distant simultaneity and all the rest).

Turning now from spacetime to the fields inscribed upon spacetime... the phrase *scalar field* refers to a single-component field which transforms by the rule

$$\varPhi(X(x)) = \varphi(x) \tag{5.0}$$

Vector fields have 4 components, and transform by the rule

$$\varPhi^\mu(X(x)) = \Lambda^\mu{}_\alpha \varphi^\alpha(x) \tag{5.1}$$

Similarly, *tensor fields* of second rank transform

$$\varPhi^{\mu\nu}(X(x)) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \varphi^{\alpha\beta}(x) \tag{5.2}$$

So it goes.

Familiarly, the electromagnetic field transforms as an *antisymmetric* tensor field of second rank:

$$\begin{aligned} F^{\mu\nu}(X(x)) &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta f^{\alpha\beta}(x) \\ f^{\alpha\beta} = -f^{\beta\alpha} \quad &\Rightarrow \quad F^{\mu\nu} = -F^{\nu\mu} \end{aligned} \tag{6}$$

We are used to deploying the elements of the field tensor as elements of an antisymmetric matrix[5]

$$\boldsymbol{f} \equiv \begin{pmatrix} 0 & - & - & - \\ f^{10} & 0 & - & f^{13} \\ f^{20} & f^{21} & 0 & - \\ f^{30} & - & f^{32} & 0 \end{pmatrix}$$

but if the independently specifiable elements are (in some arbitrary order[6]) deployed as a "6-vector" then (6) becomes

$$\begin{pmatrix} F^{10} \\ F^{20} \\ F^{30} \\ F^{32} \\ F^{13} \\ F^{21} \end{pmatrix} = \begin{pmatrix} M^{10}{}_{10} & M^{10}{}_{20} & M^{10}{}_{30} & M^{10}{}_{32} & M^{10}{}_{13} & M^{10}{}_{21} \\ M^{20}{}_{10} & M^{20}{}_{20} & M^{20}{}_{30} & M^{20}{}_{32} & M^{20}{}_{13} & M^{20}{}_{21} \\ M^{30}{}_{10} & M^{30}{}_{20} & M^{30}{}_{30} & M^{30}{}_{32} & M^{30}{}_{13} & M^{30}{}_{21} \\ M^{32}{}_{10} & M^{32}{}_{20} & M^{32}{}_{30} & M^{32}{}_{32} & M^{32}{}_{13} & M^{32}{}_{21} \\ M^{13}{}_{10} & M^{13}{}_{20} & M^{13}{}_{30} & M^{13}{}_{32} & M^{13}{}_{13} & M^{13}{}_{21} \\ M^{21}{}_{10} & M^{21}{}_{20} & M^{21}{}_{30} & M^{21}{}_{32} & M^{21}{}_{13} & M^{21}{}_{21} \end{pmatrix} \begin{pmatrix} f^{10} \\ f^{20} \\ f^{30} \\ f^{32} \\ f^{13} \\ f^{21} \end{pmatrix}$$

---

[5] See, for example, my CLASSICAL ELECTRODYNAMICS (1980), p. 162. The components $\{f^{10}, f^{20}, f^{30}\}$ are familiar as the components of the $\vec{E}$ field, and $\{f^{32}, f^{13}, f^{21}\}$ as components of the $\vec{B}$ field

[6] I select the order which in another notation reads $\{E_1, E_2, E_3, B_1, B_2, B_3\}$.

with $M^{\mu\nu}{}_{\alpha\beta} \equiv \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta - \Lambda^\mu{}_\beta\Lambda^\nu{}_\alpha$. And this invites the notational simplification

$$F^a = M^a{}_b(\mathbf{\Lambda})f^b \quad : \quad a \ \&\ b \text{ range on } \{1,2,3,4,5,6\} \tag{7}$$

We expect to be able to show that (7) is "transformationally stable," in the sense that

$$M^a{}_p(\mathbf{\Lambda}')M^p{}_b(\mathbf{\Lambda}'') = M^a{}_b(\mathbf{\Lambda}''\mathbf{\Lambda}') \tag{8}$$

Which is to say: we expect (7) to provide a 6×6 *matrix representation of the Lorentz group*. We have stepped here unwittingly onto the shore of a vast mathematical continent—"group representation theory," as in all of its intricate parts it pertains to the Lorentz group.[7]

In brief continuation of the preceding discussion: Electrodynamical theory assigns major importance to the "dual" of the field tensor $f^{\mu\nu}$, which is defined/ denoted

$$\begin{aligned} f^{\star\mu\nu} &= g^{\mu\alpha}g^{\nu\beta}f^\star{}_{\alpha\beta} \\ f^\star{}_{\alpha\beta} &\equiv \tfrac{1}{2}\epsilon_{\alpha\beta\rho\sigma}f^{\rho\sigma} \end{aligned}$$

The effect of "dualization" $f^{\mu\nu} \to f^{\star\mu\nu}$ can in $\{\vec{E},\vec{B}\}$-notation be described

$$\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_2 & -B_2 & B_1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_2 & -E_2 & E_1 & 0 \end{pmatrix}$$

of which $\{\vec{E},\vec{B}\} \to \{-\vec{B},\vec{E}\}$ serves to capture the essence. Moreover, one has[8]

$$\begin{aligned} \tfrac{1}{2}f^{\alpha\beta}f_{\beta\alpha} = -\tfrac{1}{2}f^{\star\alpha\beta}f^\star{}_{\beta\alpha} &= \vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B} \\ -\tfrac{1}{2}f^{\ \alpha\beta}f^\star{}_{\beta\alpha} &= \vec{E}\cdot\vec{B} + \vec{B}\cdot\vec{E} = 2\vec{E}\cdot\vec{B} \end{aligned}$$

---

[7] To gain a preliminary sense of the landscape, see (to select but three titles from a very long shelf) S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (1961), Chapter2; J. F. Cornwell, *Group Theory in Physics* (1984), Chapter 17; F. D. Murnaghan, *The Theory of Group Representations*,(1938), Chapter 12. Classification of the unitary representations was accomplished in this pair of classic papers: E. P. Wigner, "On unitary representations of the inhomogeneous Lorentz group," Ann. of Math. **40**, 149 (1939); V. Bargmann & E. P. Wigner, "Group theoretical discussion of relativistic wave equations," PNAS **34**, 211 (1948). In the latter connection, see also Iv. M. Shirakov, "A group-theoretical consideration of the basis of relativistic quantum mechanics. III. Irreducible representations. . . of the inhomogeneous Lorentz group," Soviet Physics JETP **6**, 929 (1958). But non-unitary representations are also of great physical importance.

[8] See pp. 256 & 298 in the notes previously cited.[5] Levi-Civita dualization is treated on pp. 163–166.

—the Lorentz invariance of which is in each case manifest. Within the 6-vector formalism we are led therefore to write

$$f = \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \rightarrow f^\star = \begin{pmatrix} -\vec{B} \\ \vec{E} \end{pmatrix}$$

and to note that we are placed thus in position to write

$$\left.\begin{aligned} \vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B} &= f^a G_{ab} f^{\ b} \\ -\vec{E}\cdot\vec{B} - \vec{B}\cdot\vec{E} &= f^a G_{ab} f^{\star b} \end{aligned}\right\} \tag{9}$$

provided we set

$$\boldsymbol{G} \equiv \|G_{ab}\| = \begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The Lorentz invariance of (9) would then follow from

$$\boldsymbol{M}^{\mathsf{T}}\boldsymbol{G}\boldsymbol{M} = \boldsymbol{G} \tag{10}$$

We may consider $\boldsymbol{G}$ to deposit "induced metric structure" on 6-space, and note that (10) is formally identical to (3). We recall finally that for electromagnetic radiation it is the case that $\vec{E}\cdot\vec{E} = \vec{B}\cdot\vec{B}$, which entails that f be "null" in the sense that $(f,f) \equiv f^a G_{ab} f^b = 0$. Returning now to the main line of this discussion...

If $\varphi(x)$ transforms as a (weightless) scalar field (which is to say: by the rule (5.0)) then

$$\partial\Phi/\partial X^\mu = \sum_\alpha (\partial x^\alpha/\partial X^\mu)(\partial\varphi/\partial x^\alpha)$$

which we express

$$\Phi_{,\mu} = \tfrac{\partial x^\alpha}{\partial X^\mu}\,\varphi_{,\alpha}$$

In words: the first partials of a scalar field transform tensorially, as components of a (weightless) covariant *vector* field.

A second differentiation gives

$$\begin{aligned} \partial^2\Phi/\partial X^\mu \partial X^\nu = \sum_{\alpha\beta} &(\partial x^\alpha/\partial X^\mu)(\partial x^\beta/\partial X^\nu)(\partial^2\varphi/\partial x^\alpha \partial x^\beta) \\ &+ \sum_\alpha (\partial^2 x^\alpha/\partial X^\mu \partial X^\nu)(\partial\varphi/\partial x^\alpha) \end{aligned}$$

Evidently we can write

$$\Phi_{,\mu\nu} = \tfrac{\partial x^\alpha}{\partial X^\mu}\tfrac{\partial x^\beta}{\partial X^\nu}\,\varphi_{,\alpha\beta}$$

and say that "the set of second partials transforms tensorially" only under conditions so special as to insure that the term $\sum_\alpha(\text{etc.})$ vanishes. And that it certainly does when the functions $x^\mu(X)$ depend only *linearly*—which is to say: *at most* linearly—upon their arguments. . . which in special relativity (inertial *Cartesian* frames interrelated by *Lorentz* transformations) is precisely the case. We conclude that, within the setting afforded by relativity, ordinary differentiation

$$\partial_\mu \text{ sends } (\text{tensor})^{\text{contravariant rank}}_{\text{covariant rank}} \longrightarrow (\text{tensor})^{\text{contravariant rank}}_{\text{covariant rank}+1}$$

This follows even if we extend the set of allowed transformations to include "inhomogeneous Lorentz transformations"[9]

$$x \to X = \boldsymbol{\Lambda} x + a$$

One fussy detail remains to be considered: if

$$X^\mu = M^\mu{}_\alpha x^\alpha$$

then

$$x^\alpha = W^\alpha{}_\nu X^\nu \quad \text{with} \quad \boldsymbol{W} = \|W^\alpha{}_\nu\| \equiv \boldsymbol{M}^{-1}$$

and to describe the transform rule for a mixed tensor we would write something like

$$\Phi^{\lambda\mu}{}_\nu = M^\lambda{}_\alpha M^\mu{}_\beta W^\gamma{}_\nu \, \varphi^{\alpha\beta}{}_\gamma$$

To underscore a presumption that $x \to X$ by *Lorentz* transformation we might specialize the notation, writing

$$X^\mu = \Lambda^\mu{}_\alpha x^\alpha$$
$$x^\alpha = V^\alpha{}_\nu X^\nu \quad \text{with} \quad \boldsymbol{V} = \|V^\alpha{}_\nu\| \equiv \boldsymbol{\Lambda}^{-1}$$

and $\Phi^{\lambda\mu}{}_\nu = \Lambda^\lambda{}_\alpha \Lambda^\mu{}_\beta V^\gamma{}_\nu \, \varphi^{\alpha\beta}{}_\gamma$. But in relativity we know something special about the construction of $\boldsymbol{V}$, for (3) supplies

$$\boldsymbol{V} \equiv \boldsymbol{\Lambda}^{-1} = \boldsymbol{g}^{-1}\boldsymbol{\Lambda}^{\mathsf{T}}\boldsymbol{g} = \|g^{\mu\alpha}\Lambda^\beta{}_\alpha g_{\beta\nu}\| = \|\Lambda_\nu{}^\mu\|$$

We can, on this basis, drop V from our list of busy symbols: the statement $\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1} = \boldsymbol{I}$ becomes $\Lambda^\mu{}_\alpha \Lambda_\nu{}^\alpha = \delta^\mu{}_\nu$, and to describe the Lorentz transform our our mixed tensor we write

$$\Phi^{\lambda\mu}{}_\nu = \Lambda^\lambda{}_\alpha \Lambda^\mu{}_\beta \Lambda_\nu{}^\gamma \, \varphi^{\alpha\beta}{}_\gamma$$

[9] Such transformations are elements of the so-called *Poincaré group*, which includes both the Lorentz group (boosts and spatial rotations) and the group of spacetime translations as subgroups.

Contraction on the last pair of indices would be accomplished

$$\begin{aligned}\Phi^{\lambda\mu}{}_{\mu} &= \Lambda^{\lambda}{}_{\alpha}\Lambda^{\mu}{}_{\beta}\Lambda_{\mu}{}^{\gamma}\,\varphi^{\alpha\beta}{}_{\gamma}\\ &= \Lambda^{\lambda}{}_{\alpha}\delta^{\gamma}{}_{\beta}\,\varphi^{\alpha\beta}{}_{\gamma}\\ &= \Lambda^{\lambda}{}_{\alpha}\,\varphi^{\alpha\beta}{}_{\beta} \quad : \quad \varphi^{\alpha\beta}{}_{\beta} \text{ transforms as a contravariant vector}\end{aligned}$$

Objects of type $\varphi^{\mu_1\mu_2\cdots\mu_r}$ possess 2^n-1 "siblings," which transform in a variety of ways, but are from a practical point of view virtually interchangeable, since the transformational distinctions are managed automatically by index placement; for example,

$$\varphi^{\mu\nu} \text{ has "siblings" } \begin{cases}\varphi_{\mu}{}^{\nu} = g_{\mu\alpha}\,\varphi^{\alpha\nu}\\ \varphi^{\mu}{}_{\nu} = g_{\nu\beta}\,\varphi^{\mu\beta}\\ \varphi_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}\,\varphi^{\alpha\beta}\end{cases}$$

In special relativity the situation acquires enhanced coherence from the circumstance that the metric structure of spacetime is at every point the same:

$$g_{\mu\nu} \text{ is } x\text{-independent} \quad : \quad g_{\mu\nu,\alpha}=0 \quad (\text{all } \mu,\nu,\alpha)$$

From this it follows that the *sibling of the derivative is the derivative of the sibling*.

Objects of type $\varphi^{\mu\nu}$ (say) respond to $x\to\mathbf{\Lambda}x$ by folding amonst themselves in $4^2=16$-dimensional representation of the Lorentz group, though

- if we impose the symmetry condition $\varphi^{\mu\nu}=\varphi^{\nu\mu}$ (which would make tensorial good sense) then only ten components are independently specifiable, and we are led to a 10-dimensional representation;
- if we impose the antisymmetry condition $\varphi^{\mu\nu}=-\varphi^{\nu\mu}$ then only six components are independently specifiable, and we are led to a 6-dimensional representation (as previously discussed).

The point I would emphasize is this: recent discussion has been tacitly specific to the *tensor representations of the Lorentz group*. There exists, however, a second broad class of representations—the so-called *spinor* representations. Corresponding statements are, for the most part, similar or identical, though there are some important differences. I reserve discussion of that aspect of our subject until we have physical reason to consider specific examples.

Principles of Lagrangian construction. The Lagrangian density $\mathcal{L}(\varphi,\partial\varphi,x)$ came first to our attention as a number-valued participant in the equation

$$S_{\mathcal{R}}[\varphi]=\tfrac{1}{c}\int_{\mathcal{R}}\mathcal{L}(\varphi,\partial\varphi,x)\;dx^0dx^1dx^2dx^3 \tag{1–18}$$

which serves to define the field-theoretic action functional. Insofar as

- complex-valued fields sometimes command our physical attention, and
- $\mathcal{L}$ sometimes enters nakedly into expressions of direct physical significance, expressions[10] which would become uninterpretable if $\mathcal{L}$ itself were complex

it becomes pertinent to stipulate that $\mathcal{L}(\varphi, \partial\varphi, x)$ will be required to be a *real*-valued function of its arguments.[11] I suspect that one could, on independent grounds, argue that $S_{\mathcal{R}}[\varphi]$ must, of necessity, be real (recall that analytic functions assume extreme values only at boundary points), and that the reality of $\mathcal{L}(\varphi, \partial\varphi, x)$ is on those grounds forced, but I do not at present know how to develop the details of such an argument.

Elementary multi-variable calculus supplies the information that integrals respond to changes of variable $x \to y = y(x)$ by the rule

$$\begin{aligned}\iint\cdots\int_{\text{bubble}} f(x)\,dx^1dx^2\cdots dx^n \\ = \iint\cdots\int_{\text{image bubble}} \underbrace{f(x(y))\left|\frac{\partial(x^1,x^2,\ldots,x^n)}{\partial(y^1,y^2,\ldots,y^n)}\right|}_{F(y)} dy^1dy^2\cdots dy^n\end{aligned}$$

The implication is that integrands transform not as simple scalars but as *scalar densities of weight* $\omega = 1$,[12] and that so, in particular, does $\mathcal{L}$. But in the context afforded by special relativity it follows from (3) that

$$\left|\frac{\partial(\;x^0,\;\;x^1,\;\;x^2,\;\;x^3)}{\partial(X^0,X^1,X^2,X^3)}\right| = \left|\frac{\partial(X^0,X^1,X^2,X^3)}{\partial(\;x^0,\;\;x^1,\;\;x^2,\;\;x^3)}\right|^{-1} = (\det\mathbf{\Lambda})^{-1} = \pm 1$$

So to the considerable extent that we can restrict our attention to "proper" Lorentz transformations (i.e., to those which are continuous with the identity, and therefore have $\det\mathbf{\Lambda} = +1$) the tensor/tensor density distinction becomes a "distinction without a difference." We will agree to file the point away as a subtlty, to be recalled when it makes a difference.

To achieve automatic Lorentz covariance (i.e., to achieve form-invariance of the field equations, and of all that follows from them) we require that $x \to \mathbf{\Lambda}x$

---

[10] The description (1–41) of the energy density provides an example.

[11] Look again, in this light, to (1–76).

[12] For discussion of the "tensor density" concept see, for example, p. 172 of the notes previously cited.[5] We have returned here to what historically was, in fact, the birthplace of that concept. Recall in this connection that it is only by virtue of the presumption that it transforms as a density (of weight $\omega = -1$) that the Levi-Civita tensor acquires the property that

$$\epsilon_{i_1 i_2 \ldots i_n} = \text{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n\end{pmatrix}$$

holds *in all coordinate systems*.

sends

$$\mathcal{L}(\varphi, \partial\varphi, x) \to \textit{same}\text{ function of the }\textit{transformed}\text{ arguments}$$

And to achieve automatic implementation of that idea, we require that $\mathcal{L}$ be expressible as a real-valued function of the *invariants* $I_1, I_2, \ldots$ which can be assembled from

- the fields $\varphi(x)$ supplied by the system $\mathcal{S}$ under consideration;
- the first partials $\partial\varphi(x)$ of those fields;
- the universally available objects $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$

—to which list we might, in exceptional cases, also adjoin the raw spacetime coordinates x^μ. The force of the program just sketched is made most vividly evident by concrete example. Look, therefore, to the field system $\mathcal{S}$ in which the players are a vector field A_ν and a tensor field $B_{\rho\sigma}$; by examination—tinkering: I know of no more systematic procedure, though doubtless one could be devised (the problem is combinatorial/graph-theoretic)—we are led to

$$\left.\begin{aligned}
I_1 &= \partial_\alpha A^\alpha\\
I_2 &= B_\alpha{}^\alpha
\end{aligned}\right\}\quad\text{linear in the fields}$$

$$\left.\begin{aligned}
I_3 &= A_\alpha A^\alpha\\
I_4 &= (\partial^\alpha A^\beta)(\partial_\alpha A_\beta)\\
I_5 &= (\partial^\alpha A^\beta)(\partial_\beta A_\alpha)\\
I_6 &= \epsilon_{\alpha\beta\rho\sigma}(\partial^\alpha A^\beta)(\partial^\rho A^\sigma)\\
I_7 &= (\partial_\alpha A_\beta)B^{\alpha\beta}\\
I_8 &= (\partial_\alpha A_\beta)B^{\beta\alpha}\\
I_9 &= A_\alpha(\partial_\beta B^{\alpha\beta})\\
I_{10} &= A_\alpha(\partial_\beta B^{\beta\alpha})\\
I_{11} &= (\partial_\alpha B^{\alpha\rho})(\partial_\beta B^\beta{}_\rho)\\
I_{12} &= (\partial_\alpha B^{\alpha\rho})(\partial_\beta B_\rho{}^\beta)\\
I_{13} &= (\partial_\alpha B^{\rho\alpha})(\partial_\beta B_\rho{}^\beta)\\
I_{14} &= (\partial_\alpha B^{\alpha\rho})(\partial_\rho B_\beta{}^\beta)\\
I_{15} &= (\partial_\alpha B^{\rho\alpha})(\partial_\rho B_\beta{}^\beta)\\
I_{16} &= (\partial_\rho B^{\alpha\beta})(\partial^\rho B_{\alpha\beta})\\
I_{17} &= (\partial_\rho B^{\alpha\beta})(\partial^\rho B_{\beta\alpha})\\
I_{18} &= \epsilon_{\alpha\beta\rho\sigma}(\partial^\alpha A^\beta)(\partial^\rho A^\sigma)\\
I_{19} &= \epsilon_{\alpha\beta\rho\sigma}B^{\alpha\beta}B^{\rho\sigma}\\
I_{20} &= \epsilon_{\alpha\beta\rho\sigma}(\partial^\alpha A^\beta)B^{\rho\sigma}\\
I_{21} &= \epsilon_{\alpha\beta\rho\sigma}A^\alpha(\partial^\beta B^{\rho\sigma})
\end{aligned}\right\}\quad\text{quadratic in the fields}$$

$$\left.I_{22} = A_\alpha A_\beta B^{\alpha\beta}\right\}\quad\text{cubic in the fields}$$

I may have missed some, but you get the idea. Higher order invariants factor into products of those listed. The list would be shortened if (anti)symmetry properties were imposed upon $B_{\rho\sigma}$, much lengthened if either the rank or the number of the participating fields were increased. It would be lengthened also if we allowed ourselves to introduce translational symmetry-breaking terms such as $I_{21} = x^\alpha A_\alpha$.

Relativistically covariant accounts of the dynamics of such systems would now result from writing

$$\mathcal{L}(A, B, \partial A, \partial B) = \ell(I_1, I_2, \ldots, I_{22})$$

But what functional form should be assigned to ℓ(etc.)? As physicists it has been our reductionistic practice—a practice which has enjoyed a high degree of success, but which is supported no physically/philosophically secure foundation, and which in this wholistic world may be doomed to ultimate failure—to resolve systems into their imagined "component parts," and then to study how those parts interact. It becomes in this light natural to write

$$\mathcal{L}(A, B, \partial A, \partial B) = \mathcal{L}_A(A, \partial A) + \mathcal{L}_B(B, \partial B) + g \cdot \mathcal{L}_{\text{interaction}}(A, B, \partial A, \partial B)$$

where $\mathcal{L}_A(A, \partial A)$ describes the dynamics of "free A-fields," $\mathcal{L}_B(B, \partial B)$ the dynamics of "free B-fields," and the control parameter g describes the strength of the (typically weak) interaction. But we are brought thus only part way to the resolution of our problem, for. . .

When we look (say) to $\mathcal{L}_A(A, \partial A) = \ell_A(I_1, I_3, I_4)$ we still confront this question: What functional structure should we assign to ℓ_A(etc.)? Here major simplification results, and some important physics is brought into view, if we declare a special interest in *linear* field theories—theories dominated by a *principle of superposition*. For then $\mathcal{L}_A(A, \partial A)$ has necessarily to be *quadratic* in its arguments,[13] and the number of available options is greatly reduced: we have

$$\begin{aligned}
\mathcal{L}_A(A, \partial A) &= c_4 I_4 + c_5 I_5 + c_6 I_6 + c_1 I_1^2 + c_3 I_3 \\
&= c_4 (\partial^\alpha A^\beta)(\partial_\alpha A_\beta) + c_5 (\partial^\alpha A^\beta)(\partial_\beta A_\alpha) + c_6\, \epsilon_{\alpha\beta\rho\sigma}(\partial^\alpha A^\beta)(\partial^\rho A^\sigma) \\
&\qquad\qquad + c_1 (\partial_\alpha A^\alpha)(\partial_\beta A^\beta) + c_3 A_\alpha A^\alpha
\end{aligned}$$

where $c_1, c_3, \ldots, c_6$ are adjustable constants. By a similar argument

$$\mathcal{L}_B(B, \partial B) = \text{linear combination of } I_{11}, I_{12}, I_{13}, I_{14}, I_{15}, I_{16}, I_{17}, I_{19} \text{ and } I_2^2$$

[13] More formally, it has to be *homogeneous of degree 2:*

$$\mathcal{L}_A(\lambda A, \lambda \partial A) = \lambda^2 \mathcal{L}_A(A, \partial A)$$

It is with less conviction that we would assert $\mathcal{L}_{\text{interaction}}$ to be a linear combination of $I_7, I_8, I_9, I_{10}, I_{20}$ and I_{21}, for interaction (external forcing) typically entails a suspension of the principle of superpositon. Recall the situation in electrodynamics, where

$$\partial_\mu F^{\mu\nu} = \begin{cases} 0 & \text{in the absence of sources} \\ J^\mu & \text{in the presence of sources} \end{cases}$$

Solutions of the former equation yield other solutions when added, solutions of the latter equation do not (though to any particular solution of the latter one can add arbitrary solutions of the former).

To summarize: Special relativity (acting conjointly with the principle of superposition) exerts a fairly strong constraint upon the design of free field theories, but to achieve a convincing "theory of interaction" we need the input of a new idea.[14]

One final remark before we turn to discussion of some particular free field theories: Such structural properties as we may impute to $\mathcal{L}$ are susceptible to seeming contravention by gauge transformation

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \partial_\alpha \mathcal{G}^\alpha \tag{1–23}$$

For $\mathcal{G}^\alpha$—which can be "anything"—is free to violate just about any condition we may have in mind.

Real scalar field: the Klein-Gordon equation. Suppose $\mathcal{S}$ affords only a single real-valued scalar field $\varphi(x)$. The quadratic invariants which can be assembled from φ and $\partial_\mu\varphi$ are two in number:

$$I_1 = (\partial^\alpha\varphi)(\partial_\alpha\varphi) \quad \text{and} \quad I_2 = \varphi^2$$

so we are led to write

$$\begin{aligned} \mathcal{L}(\varphi, \partial\varphi) &= c_1 I_1 + c_2 I_2 \\ &= \tfrac{1}{2}K\Big\{g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} - \varkappa^2\varphi^2\Big\} \end{aligned} \tag{11}$$

where $[K] = (\text{energy/length})/[\varphi]^2$ and $[\varkappa^2] = 1/(\text{length})^2$. The resulting field equation

$$\Big\{\partial_\mu \frac{\partial}{\partial\varphi_{,\mu}} - \frac{\partial}{\partial\varphi}\Big\}\mathcal{L} = \partial_\mu(g^{\mu\alpha}\varphi_{,\alpha}) + \varkappa^2\varphi = 0$$

---

[14] It was partly to achieve the latter objective that *gauge field theory* was invented; see the introduction to Chapter 9 in L. O'Raifeartaigh's *The Dawning of Gauge Theory* (1997), which alludes to the motivational pulse of Ronald Shaw's doctoral research (1952–54). Shaw's thesis joins a classic paper by C. N. Yang & R. Mills ("Isotopic spin conservation and a generalized gauge invariance," PR **95**, 631 (1954)) as one of the historic sources of gauge theory in its modern form.

can be written

$$\big(\square + \varkappa^2\big)\varphi = 0 \tag{12}$$

where

$$\square \equiv g^{\alpha\beta}\partial_\alpha\partial_\beta = \tfrac{1}{c^2}\big(\tfrac{\partial}{\partial t}\big)^2 - \nabla^2 \tag{13}$$

defines the *wave operator* (or "d'Alembertian").

At (12) we have encountered the celebrated *Klein-Gordon equation*, though the equation studied by O. Klein and W. Gordon in 1926—which had been written down but abandoned by Schrödinger himself even prior to the invention of the non-relativistic "Schrödinger equation"—was actually the complex analog of (12). Useful insight into the historical origins of (12) is gained when one sets

$$\varphi(x) = e^{\frac{i}{\hbar}(p,x)} \quad : \quad (p,x) \equiv p_\alpha x^\alpha = Et - \boldsymbol{p}\cdot\boldsymbol{x}$$

and finds that (12) will be satisfied if and only if

$$g^{\alpha\beta}p_\alpha p_\beta \equiv (E/c)^2 - \boldsymbol{p}\cdot\boldsymbol{p} = \hbar^2\varkappa^2$$

But the relativistic theory of a mass point m presents us with the "dispersion relation" $(E/c)^2 - \boldsymbol{p}\cdot\boldsymbol{p} = (mc)^2$. It becomes in this light natural to set

$$\varkappa = \frac{mc}{\hbar} \tag{14}$$

and to consider the Klein-Gordon equation (12) to be fruit of the "Schrödinger quantization" procedure

$$g^{\alpha\beta}p_\alpha p_\beta\Big|_{p\to i\hbar\partial} = (mc)^2$$

Notice that the Klein-Gordon equation gives back the wave equation $\square\varphi = 0$ in the limit $\varkappa \downarrow 0$; i.e., in the limit of zero mass.[15]

Working from (1–34) we find the stress-energy tensor of the real scalar field to be given by

$$\begin{aligned} S^\mu{}_\nu &= \frac{\partial\mathcal{L}}{\partial\varphi_{,\mu}}\,\varphi_{,\nu} - \mathcal{L}\delta^\mu{}_\nu \qquad (15)\\ &= \tfrac{1}{2}K\Big\{(g^{\mu\alpha} + g^{\alpha\mu})\varphi_{,\alpha}\varphi_{,\nu} - (g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta})\delta^\mu{}_\nu + \varkappa^2\varphi^2\delta^\mu{}_\nu\Big\} \end{aligned}$$

[15] For more detailed discussion of the Klein-Gordon equation, see Chapter 3 in Schweber[7] or Chapter 20 §5 in A. Massiah, *Quantum Mechanics* (1966). Schrödinger abandoned his relativistic equation for the good physical reason that it led to the wrong hydrogen spectrum: see §51 in L. I. Schiff, *Quantum Mechanics* (3rd edition 1968) for details and references.

from which we learn (see again (1–41)) that the *energy density* of such a field can be described

$$\begin{aligned}\mathcal{E} = S^0{}_0 &= \tfrac{1}{2}K\Big\{2(\varphi_{,0})^2 - \big[(\varphi_{,0})^2 - (\varphi_{,1})^2 - (\varphi_{,2})^2 - (\varphi_{,3})^2\big] + \varkappa^2\varphi^2\Big\}\\ &= \tfrac{1}{2}K\Big\{(\varphi_{,0})^2 + (\varphi_{,1})^2 + (\varphi_{,2})^2 + (\varphi_{,3})^2 + \varkappa^2\varphi^2\Big\} \qquad (16.1)\end{aligned}$$

It was in anticipation of this result—i.e., to ensure the non-negativity of energy density—that we gave the name $\varkappa^2$ to the coefficient of the φ^2-term in (11). The three components of momentum density are given by

$$c\mathcal{P}_i = S^0{}_i = \frac{\partial\mathcal{L}}{\partial\varphi_{,0}}\,\varphi_{,\nu} = K\varphi_{,0}\varphi_{,i} \quad : \quad (i = 1, 2, 3) \qquad (16.2)$$

Complex scalar field. Suppose $\mathcal{S}$ affords only a single *complex*-valued scalar field

$$\begin{aligned}\psi\ (x) &= \varphi_1(x) + i\varphi_2(x) = A(x)e^{+i\phi(x)}\\ \psi^*(x) &= \varphi_1(x) - i\varphi_2(x) = A(x)e^{-i\phi(x)}\end{aligned}$$

The quadratic invariants which can be assembled from ψ, ψ^*, $\partial_\mu\psi$ and $\partial_\mu\psi^*$ are quickly listed, and lead one to contemplate Lagrange densities of the form

$$\mathcal{L} = \tfrac{1}{2}g^{\alpha\beta}\big\{a\,\psi_{,\alpha}\psi_{,\beta} + 2a_0\psi^*_{,\alpha}\psi_{,\beta} + a^*\psi^*_{,\alpha}\psi^*_{,\beta}\big\} + \tfrac{1}{2}\big\{b\,\psi\psi + 2b_0\psi^*\psi + b^*\psi^*\psi^*\big\}$$

where the postulated reality condition $\mathcal{L} = \mathcal{L}^*$ requires that a_2 and b_2 be real. If we allow ourselves to write

$$\boldsymbol{\psi} \equiv \begin{pmatrix}\psi\\ \psi^*\end{pmatrix} \quad \text{and} \quad \boldsymbol{\psi}^\dagger \equiv \begin{pmatrix}\psi^*\\ \psi\end{pmatrix}^{\mathsf{T}} = \begin{pmatrix}\psi^* & \psi\end{pmatrix}$$

and to drop the commas which signify differentiation (writing $\boldsymbol{\psi}_\mu \equiv \partial_\mu\boldsymbol{\psi}$) then the preceding $\mathcal{L}$ can be displayed in the following somewhat more orderly manner:

$$\mathcal{L} = \tfrac{1}{2}g^{\alpha\beta}\boldsymbol{\psi}^\dagger_\alpha\begin{pmatrix}a_0 & a^*\\ a & a_0\end{pmatrix}\boldsymbol{\psi}_\beta + \tfrac{1}{2}\boldsymbol{\psi}^\dagger\begin{pmatrix}b_0 & b^*\\ b & b_0\end{pmatrix}\boldsymbol{\psi}$$

The reality condition can be said in this notation to result from the *hermiticity* of the 2×2 matrices. Reality would be retained even if we were to relax the requirement that the diagonal elements be equal, but when we do so—writing

$$\mathcal{L} = \tfrac{1}{2}g^{\alpha\beta}\boldsymbol{\psi}^\dagger_\alpha\,\mathbb{A}\,\boldsymbol{\psi}_\beta + \tfrac{1}{2}\boldsymbol{\psi}^\dagger\mathbb{B}\,\boldsymbol{\psi} \qquad (17)$$

with

$$\mathbb{A} = \begin{pmatrix}a_0 + a_3 & a_1 - ia_2\\ a_1 + ia_2 & a_0 - a_3\end{pmatrix} \quad \text{and} \quad \mathbb{B} = \begin{pmatrix}b_0 + b_3 & b_1 - ib_2\\ b_1 + ib_2 & b_0 - b_3\end{pmatrix}$$

—it becomes apparent that the new parameters a_3 and b_3 actually make no net contribution to $\mathcal{L}$, therefore none to the field equations, none to the physics. In (17) we have, therefore, what is in effect a 6-parameter population of theories.

In an effort to sharpen the problem before us (and to prepare the ground for some future work), we—drawing inspiration from quantum mechanics—agree to look only to those theories which possess

$$\left.\begin{array}{l}\psi \ \ \to e^{+i\omega}\psi\\ \psi^* \to e^{-i\omega}\psi^*\end{array}\right\} \quad : \quad \text{equivalently} \quad \boldsymbol{\psi} \to \begin{pmatrix} e^{+i\omega} & 0\\ 0 & e^{-i\omega}\end{pmatrix}\boldsymbol{\psi} \tag{18}$$

as an internal symmetry. Immediately $a = b = 0$ ($\mathbb{A}$ and $\mathbb{B}$ must, in other words, be diagonal—effectively multiples of the identity), and we a Langrangian which, after notational adjustments, can be expressed

$$\mathcal{L} = \tfrac{1}{2}K\Big\{g^{\alpha\beta}\psi^*_{,\alpha}\psi_{,\beta} - \varkappa^2\psi^*\psi\Big\} \tag{19}$$

This, significantly, is the most general instance of our original Lagrangian which manifests the property that it is *bilinear* in starred and unstarred field variables.

The resulting field equations read

$$\left.\begin{array}{l}\Big\{\partial_\mu\dfrac{\partial}{\partial\psi_{,\mu}} - \dfrac{\partial}{\partial\psi}\ \Big\}\mathcal{L} = \partial_\mu(g^{\mu\alpha}\psi^*_{,\alpha}) + \varkappa^2\psi^* = \big(\Box + \varkappa^2\big)\psi^* = 0\\ \Big\{\partial_\mu\dfrac{\partial}{\partial\psi^*_{,\mu}} - \dfrac{\partial}{\partial\psi^*}\Big\}\mathcal{L} = \partial_\mu(g^{\mu\alpha}\psi_{,\alpha}) + \varkappa^2\psi \ \ = \big(\Box + \varkappa^2\big)\psi \ \ = 0\end{array}\right\} \tag{20}$$

Bilinearity has had the consequence that
- ψ -variation yields a field equation involving only ψ^*;
- ψ^*-variation yields a field equation involving only ψ :

the equations are uncoupled; each is the complex conjugate of the other.

The stress-energy tensor of any (ψ,ψ^*)-field system can be described

$$S^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial\psi_{,\mu}}\,\psi_{,\nu} + \frac{\partial\mathcal{L}}{\partial\psi^*_{,\mu}}\,\psi^*_{,\nu} - \mathcal{L}\delta^\mu{}_\nu$$

which for systems of type (19) becomes

$$S^\mu{}_\nu = \tfrac{1}{2}K\Big\{g^{\mu\alpha}\big(\psi^*_{,\alpha}\psi_{,\nu} + \psi_{,\alpha}\psi^*_{,\nu}\big) - \big(g^{\alpha\beta}\psi^*_{,\alpha}\psi_{,\beta}\big)\delta^\mu{}_\nu + \varkappa^2\psi^*\psi\,\delta^\mu{}_\nu\Big\}$$

That—in consequence ultimately of the field equations—the components of the stress-energy tensor enter into a quartet of conservation laws

$$\partial_\mu S^\mu{}_\nu = 0$$

we know already on quite general grounds; those statements follow[16] from the circumstance that x does not enter *explicitly* into the design of $\mathcal{L}$: $\boldsymbol{\partial}_\mu\mathcal{L} = 0$. Energy density, for systems of type (19), can be described

$$\mathcal{E} = S^0{}_0 = \tfrac{1}{2}K\Big\{\psi^*_{,0}\psi_{,0} + \psi^*_{,1}\psi_{,1} + \psi^*_{,2}\psi_{,2} + \psi^*_{,3}\psi_{,3} + \varkappa^2\psi^*\psi\Big\} \geqslant 0$$

[16] See again the discussion subsequent to (1–34).

The results just summarized derive from the translational map. For more general maps, Noether's theorem—adapted to systems of type (19)—supplies

$$J^\mu_r = \tfrac{1}{2}K\Big\{g^{\mu\alpha}\psi^*_{,\alpha}\big[\Phi_r - \psi_{,\beta}\mathcal{X}^\beta_r\big] + g^{\mu\alpha}\psi_{,\alpha}\big[\Phi^*_r - \psi^*_{,\beta}\mathcal{X}^\beta_r\big] + \big[g^{\alpha\beta}\psi^*_{,\alpha}\psi_{,\beta} - \varkappa^2\psi^*\psi\big]\mathcal{X}^\mu_r\Big\}$$

For the one-parameter internal symmetry map (18) the r-subscript can be abandoned, and we have

$$\Phi = i\psi \quad\text{and}\quad \mathcal{X}^\mu = 0$$

giving[17]

$$Q^\mu = i\,\tfrac{1}{2}Kg^{\mu\alpha}\Big\{\psi^*_{,\alpha}\psi - \psi^*\psi_{,\alpha}\Big\} \tag{21}$$

We expect to have $\partial_\mu Q^\mu = 0$, and by calculation

$$\begin{aligned}\partial_\mu Q^\mu &= i\,\tfrac{1}{2}Kg^{\mu\alpha}\Big\{\psi^*_{,\mu\alpha}\psi + \psi^*_{,\alpha}\psi_{,\mu} - \psi^*_{,\mu}\psi_{,\alpha} - \psi^*\psi_{,\mu\alpha}\Big\}\\ &= i\,\tfrac{1}{2}K\Big\{\psi\,\Box\,\psi^* - \psi^*\,\Box\,\psi\Big\} \quad\text{after a cancellation}\\ &= -i\,\tfrac{1}{2}K\Big\{\psi\,\varkappa^2\,\psi^* - \psi^*\varkappa^2\,\psi\Big\} \quad\text{by the field equations}\\ &= 0\end{aligned}$$

find this to be in fact the case *by implication of the equations of motion*.

Real vector field: Procca's equations. Suppose system $\mathcal{S}$ affords only a solitary real-valued vector field $U_\mu(x)$. The quadratic invariants which can be assembled from U and $\partial_\mu U$ are (see again the list on p. 10) five in number, and lead one to contemplate Lagrangians[18] of the form

$$\mathcal{L} = \tfrac{1}{2}c_0U^\alpha U_\alpha + \tfrac{1}{2}\big\{c_1g^{\alpha\rho}g^{\sigma\beta} + c_2g^{\alpha\sigma}g^{\beta\rho} + c_3g^{\alpha\beta}g^{\rho\sigma} + c_4\epsilon^{\alpha\beta\rho\sigma}\big\}U_{\alpha,\beta}U_{\rho,\sigma}$$

The associated field equations can be written

$$\partial_\nu\big\{c_1U^{\mu,\nu} + c_2U^{\nu,\mu} + c_3g^{\mu\nu}U^\alpha{}_{,\alpha}\big\} - c_0U^\mu = 0$$

giving

$$c_1\Box U_\mu + (c_2 + c_3)U^\alpha{}_{,\alpha\mu} - c_0U_\mu = 0$$

[17] The notation J^μ_r is generic; typically one adopts a non-generic notation to reflect the fact that one has been led from a specialized map to an object to which one intends to give a name more specific than "Noetherean current." Here my Q^μ is intended to as an allusion to "charge."

[18] I will hereafter allow myself—in the company of the rest of the world, and except when confusion might result (which is seldom)—to say "Lagrangian" when I mean "Lagrange density."

The surprising absence of c_4 from the preceding equation can be understood as follows: define $\mathcal{G}^\sigma \equiv \epsilon^{\alpha\beta\rho\sigma}U_{\alpha,\beta}U_\rho$ and observe that

$$\partial_\sigma \mathcal{G}^\sigma = \epsilon^{\alpha\beta\rho\sigma}U_{\alpha,\beta}U_{\rho,\sigma} + \underbrace{\epsilon^{\alpha\beta\rho\sigma}U_{\alpha,\beta\sigma}U_\rho}_{0 \text{ by the } \beta\sigma\text{-symmetry of } U_{\alpha,\beta\sigma}}$$

The c_4-term is absent from the equations of motion because it is a gauge term (can be "gauged away"). Evidently we lose nothing if we write

$$\mathcal{L} = \tfrac{1}{2}\big\{c_1 g^{\alpha\rho}g^{\sigma\beta} + c_2 g^{\alpha\sigma}g^{\beta\rho} + c_3 g^{\alpha\beta}g^{\rho\sigma}\big\}U_{\alpha,\beta}U_{\rho,\sigma} - \tfrac{1}{2}\varkappa^2 U^\alpha U_\alpha \tag{22}$$

which gives

$$c_1 \Box U_\mu + (c_2 + c_3)\partial_\mu(\partial_\alpha U^\alpha) + \varkappa^2 U_\mu = 0 \tag{23}$$

We have now in hand a 4-parameter population of vector field theories. How are we to isolate the most interesting specimens? Classical electrodynamics provides the vector field with which we have greatest familiarity, and it is to electrodynamics we will look for guidance. The effort will serve to lend deepened comprehension to what kind of a thing it is that Maxwell gave to the world.

The vector field of electrodynamical interest is the so-called "4-potential" $A^\mu(x)$, introduced via

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{24.1}$$

in order to render the sourceless half

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \tag{24.2}$$

of Maxwell's equations automatic. The sourcey other half

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{24.3}$$

then become

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu \tag{24.4}$$

from which charge conservation

$$\partial_\nu J^\nu = 0 \tag{24.5}$$

follows as an immediate corollary. Potentials A_μ and $A'_\mu = A_\mu + \partial_\mu\chi$ (with χ arbitrary) are "gauge equivalent" in the sense that both, when introducted into the right side of (24.1), yield the same $F^{\mu\nu}$. That is why the A^μ-field is called a "potential," and denied any direct claim to physicality;[19] "physicality" might be attributed to the *set* $\{A^\mu\}$ of gauge-equivalent 4-potentials, or to any

[19] The story is a familiar one; it is for an identical reason that "direct physicality" cannot be attributed to the Lagrangian $\mathcal{L}$. The spooks appear to be in control!

gauge-invariant attribute of that set, but not to its individual elements. Pick an arbitrary element A'_μ of the set, write $A_\mu = A'_\mu - \partial_\mu\chi$ and select χ to be any solution of $\Box\chi = \partial^\mu A'_\mu$; one has then achieved

$$\partial_\mu A^\mu = 0 \tag{24.6}$$

This is the "Lorentz gauge condition," which can without loss of generality always be *assumed* to pertain. When that assumption is in force one can, in place of (24.4), write the simpler equation

$$\Box A^\nu = J^\nu \tag{24.7}$$

In the absence of sources (i.e., for *free* electromagnetic fields) the right sides of (24.3/4/7) vanish, and all else remains unchanged. In particular, (23.4) becomes

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0 \tag{24.8}$$

Note that we have been brought thus far by Maxwell, without any reference to the resources of Lagrangian field theory.

The field equations (23) would assume the electromagnetic form (24.8) were we to set $\varkappa = 0$ and $c_2 + c_3 = -c_1$; were we, in other words, to set

$$c_1 + c_2 + c_3 = 0 \quad \text{and} \quad \varkappa = 0$$

Look now to the class of theories which results when one retains the former of those conditions but drops the latter:

$$c_1 + c_2 + c_3 = 0 \quad \text{and} \quad \varkappa \neq 0$$

The field equations (23) then read

$$c_1\big\{\Box U_\mu - \partial_\mu(\partial_\alpha U^\alpha)\big\} + \varkappa^2 U_\mu = 0 \tag{25}$$

which when hit with ∂^μ yield the "Lorentz gauge-like" statement

$$\partial_\mu U^\mu = 0 \tag{26}$$

not as an arbitrarily imposed side condition but as an *enforced corollary of the equations of motion*. Returning with this information to (25) we obtain

$$\big(\Box + \varkappa^2\big)U^\mu = 0 \tag{27}$$

The solutions of $c_1 + c_2 + c_3 = 0$ can, for given c_1, be parameterized

$$\left.\begin{aligned} c_2 &= +\lambda c_1 - c_1 \\ c_3 &= -\lambda c_1 \end{aligned}\right\} \quad : \quad \lambda \text{ arbitrary}$$

and the Lagrangian (22) in this notation becomes[20]

$$
\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}\big\{g^{\alpha\rho}g^{\sigma\beta} - (1-\lambda)g^{\alpha\sigma}g^{\beta\rho} - \lambda g^{\alpha\beta}g^{\rho\sigma}\big\}U_{\alpha,\beta}U_{\rho,\sigma} - \tfrac{1}{2}\varkappa^2 U^\alpha U_\alpha \\
&= \tfrac{1}{2}U^{\alpha,\beta}(U_{\alpha,\beta} - U_{\beta.\alpha}) - \tfrac{1}{2}\varkappa^2 U^\alpha U_\alpha \\
&\qquad + \lambda \underbrace{\big\{U^\alpha{}_{,\beta}U^\beta{}_{,\alpha} - U^\alpha{}_{,\alpha}U^\beta{}_{,\beta}\big\}}_{=\,\partial_\alpha\{U^\alpha{}_{,\beta}U^\beta - U^\alpha U^\beta{}_{,\beta}\}}
\end{aligned}
$$

We might as well discard the λ-term, since it can (as indicated) always be gauged away; this done, we have

$$\mathcal{L} = \tfrac{1}{2}U^{\alpha,\beta}(U_{\alpha,\beta} - U_{\beta,\alpha}) - \tfrac{1}{2}\varkappa^2 U^\alpha U_\alpha \tag{28.1}$$

But $U^{\alpha,\beta} = \frac{1}{2}\big(U^{\alpha,\beta} + U^{\beta,\alpha}\big) + \frac{1}{2}\big(U^{\alpha,\beta} - U^{\beta,\alpha}\big) = U^{\alpha,\beta}_{\text{symmetric}} + U^{\alpha,\beta}_{\text{antisymmetric}}$ and only the latter term survives the double summation process; it becomes therefore possible to write

$$\mathcal{L} = \tfrac{1}{4}(U^{\alpha,\beta} - U^{\beta,\alpha})(U_{\alpha,\beta} - U_{\beta,\alpha}) - \tfrac{1}{2}\varkappa^2 U^\alpha U_\alpha \tag{28.2}$$

Proceeding in imitation of our electrodynamical experience, we introduce

$$G^{\mu\nu} \equiv \partial^\mu U^\nu - \partial^\nu U^\mu = -\big(U^{\mu,\nu} - U^{\nu,\mu}\big) \tag{29}$$

from which

$$\partial^\lambda G^{\mu\nu} + \partial^\mu G^{\nu\lambda} + \partial^\nu G^{\lambda\mu} = 0 \tag{30}$$

follows as a corollary. Notice that if (30) were—as in electrodynamics—*postulated*, then the possibility of writing (29) would be implied,[21] but U^μ would be determined *only up to gauge*, and the "Lorentz gauge-like" condition (26) would not be automatically in force. Recall also that it *is* in force only because we have assumed $\varkappa \neq 0$.

The road just travelled was first travelled by A. Procca,[22] who sought to create what was in effect a "theory of massive photons." In the following figure

---

[20] To simplify ensuing discussion we at this point set $c_1 = 1$; i.e., we absorb c_1 into the definitions of U^μ and $\varkappa$.

[21] Analogously: if it is known of $\boldsymbol{F}$ that $\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{0}$, then $\boldsymbol{F} = -\boldsymbol{\nabla}U$ is implied, but U is determined only up to gauge: $U \to U' = U + \text{constant}$, and must therefore be considered "unphysical." But if we take U—considered "physical" (no gauge)—to be our point of departure and introduce $\boldsymbol{F} \equiv -\boldsymbol{\nabla}U$ as a *definition*, then $\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{0}$ acquires the status of an "interesting corollary."

[22] "Sur la Théorie Ondulatoire des Electrons Positifs et Négatifs," Jour. Phys. Rad. **7**, 347 (1936). Procca, pursuing a fundamental idea injected into physics the previous year by H. Yukawa, sought to create a "theory of vector mesons" which adhered as closely as possible to the electromagnetic model. The subject is explored in much greated detail in my RELATIVISTIC CLASSICAL FIELDS (1973), pp. 85–97. My principal sources were W. Pauli, "Relativistic field theories of elementary particles," Rev. Mod. Phys. **13**, 203 (1941) Part II §2; E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (1953), pp. 99–103; G. Wentzel, *Quantum Theory of Fields* (1949), Chapter 3.

FREE MAXWELL FIELD	FREE PROCCA FIELD
$\bullet\ \partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$	$\partial^\lambda G^{\mu\nu} + \partial^\mu G^{\nu\lambda} + \partial^\nu G^{\lambda\mu} = 0$
$\Downarrow$	$\partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu = 0$
$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$	$\Uparrow$
$\bullet\quad \partial_\mu F^{\mu\nu} = 0$	$G^{\mu\nu} \equiv \partial^\mu U^\nu - \partial^\nu U^\mu$
$\Downarrow$	$\uparrow$
$\square A^\nu = 0$ in Lorentz gauge:	$\bullet\ \big(\square + \varkappa^2\big)U^\nu = 0$
$\partial_\mu A^\mu = 0$	$\bullet\qquad \partial_\mu U^\mu = 0$

I have attempted to display/contrast the logical pattern of the two theories in question. I have used $\bullet$ to mark the postulated free field equations, and arrows to indicate the flow of subsequent argument. Note that while in Maxwellean theory $F^{\mu\nu}$ is physical/fundamental, and A^μ an auxiliary construct, in Procca's theory it is the vector field U^μ which is fundamental, and the tensor field which plays the auxiliary role. Comparison of the two theories puts one in position to assert that *gauge transformations*

$$A_\mu \to A'_\mu = A + \partial_\mu \chi$$

enter into electromagnetic discourse in consequence of the "masslessness of the photon:" $\varkappa = 0$*.* In the contrary case ($\varkappa \neq 0$) the Lorentz gauge becomes not an option, but a mandate.

Procca has given us an good, unexceptionable-in-every-way example of a Lagrangian field theory. Actually a class of such examples...which have some surprising things to teach us. Notice first that if (28.2) is renotated

$$\mathcal{L} = \tfrac{1}{2}\Big\{G^{\sigma\rho}(U_{\rho,\sigma} - U_{\sigma,\rho}) - \tfrac{1}{2}G^{\rho\sigma}G_{\rho\sigma}\Big\} - \tfrac{1}{2}\varkappa^2 U^\rho U_\rho \tag{31}$$

and if U^μ and $G^{\mu\nu} = -G^{\nu\mu}$ are construed to be *independent* fields (i.e., if we suppose ourselves to be considering the interaction of a vector field with an antisymmetric tensor field), then construction of the field equations

$$\begin{aligned}
\Big\{\partial_\mu \frac{\partial}{\partial G_{\alpha\beta,\mu}} - \frac{\partial}{\partial G_{\alpha\beta}}\Big\}\mathcal{L} &= -\tfrac{1}{2}\Big\{\big(U^{\beta,\alpha} - U^{\alpha,\beta}\big) - G^{\alpha\beta}\Big\} = 0\\
&\Downarrow\\
G^{\mu\nu} &= \partial^\mu U^\nu - \partial^\nu U^\mu && (32.1)\\
&\Downarrow\\
\therefore\quad \partial^\lambda G^{\mu\nu} + \partial^\mu G^{\nu\lambda} + \partial^\nu G^{\lambda\mu} &= 0 && (32.2)
\end{aligned}$$

$$\begin{aligned}
\Big\{\partial_\mu \frac{\partial}{\partial U_{\alpha,\mu}} - \frac{\partial}{\partial U_\alpha}\Big\}\mathcal{L} &= \partial_\mu \tfrac{1}{2}\big(G^{\mu\alpha} - G^{\alpha\mu}\big) + \varkappa^2 U^\alpha = 0\\
&\Downarrow\\
\partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu &= 0 \quad \text{by } G^{\mu\nu}\text{-antisymmetry} && (32.3)\\
&\Downarrow\\
\therefore\quad \partial_\nu U^\nu &= 0 && (32.4)
\end{aligned}$$

is found to yield the entire set of Procca equations (including the equation (32.1) which previously served to *define* $G^{\mu\nu}$).

We touch here on a formal problem which causes electrodynamics—historic mother of classical field theory though she is—to stand somewhat apart from the main line of field-theoretic development. . . as an "exceptional case." If in (32) we set $\varkappa \to 0$ then we obtain equations which differ only notationally[23] from the free-field equations of Maxwellian theory (in Lorentz gauge). But if *in the Lagrangian* (31) we set $\varkappa = 0$ then we lose the leverage which gave us (32.4); we obtain the equations of free-field electrodynamics *except for the Lorentz gauge condition*, in the absence of which the remaining equations are incorrect, but which must be brought into the theory as an unmotivated import. A. O. Barut, on p. 102 of his *Electrodynamics and Classical Theory of Fields & Particles* (1964), tabulates four different "electromagnetic free-field Lagrangians"—one advocated by Fermi, another by Schwinger, but all of which suffer from the formal defect just described.[24] One can temper the problem by adopting Procca theory (i.e., by pretending that "the photon has mass") for the purposes of calculation, and then "turning off $\varkappa$" at the end of the day.

One final remark, intended to clarify the "theoretical placement" of the preceding discussion, the distinction between what we have accomplished and what we have not: In physical applications of 3-dimensional vector analysis (whether to electrodynamics, to fluid dynamics, . . .) one frequently gains very useful analytical leverage from *Helmholtz' theorem*, according to which every vector field $\boldsymbol{V}(x)$ can be expressed as the superposition

$$\boldsymbol{V}(x) = \boldsymbol{S}(x) + \boldsymbol{I}(x)$$

of a *solenoidal* field $\boldsymbol{S}(x)$ and an *irrotational* field $\boldsymbol{I}(x)$, where

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{S} = 0 &\Rightarrow \text{there exists an } \boldsymbol{\Omega} \text{ field such that } \boldsymbol{S} = \boldsymbol{\nabla}\times\boldsymbol{\Omega}\\ \boldsymbol{\nabla}\times\boldsymbol{I} = \boldsymbol{0} &\Rightarrow \text{there exists an } \omega \text{ field such that } \boldsymbol{I} = \boldsymbol{\nabla}\omega\end{aligned}$$

Helmholtz' theorem[25] speaks of a particular instance of the vastly more general *Hodge decomposition theorem*,[26] which—as it refers to vector fields on spacetime—asserts that every such field can be represented

$$V_\mu = S_\mu + I_\mu + V_\mu^0$$

where $\partial^\mu S_\mu = 0$, $\partial_\mu I_\nu - \partial_\nu I_\mu = 0$ and $\Box V_\mu^0 = 0$, and that the representation is unique. In the preceding discussion we have been lead from these basic assumptions

$$c_1 + c_2 + c_3 = 0 \quad \text{and} \quad \varkappa \neq 0$$

---

[23] Change $U^\mu \to A^\mu$, $G^{\mu\nu} \to F^{\mu\nu}$.

[24] See also §4–9 in F. Rohrlich, *Classical Charged Particles* (1965).

[25] See R. B. McQuistan, *Scalar and Vector Fields: A Physical Interpretation* (1965) §11.5 for a detailed proof.

[26] See H. Flanders, *Differential Forms, with Applications to the Physical Sciences* (1963), p. 138.

to Procca fields U_μ from which the "irrotational" and "harmonic" terms are absent. We would expect such terms to be called into play if our assumptions were relaxed.[27]

Introduction to the "canonical formulation" of relativistic free-field theory. The following remarks are intended to place in useful perspective—and in that sense to be preparatory for—discussion of the Dirac equation. But they are of some independent interest, and will serve to indicate one of the portals through which algebra and group representation theory enter into field-theoretic discourse.

Let $m\ddot{x} = -U'(x)$ speak for the vast population of second-order differential equations which Nature dumps upon us in such variety. Through each initial point $x(0)$ pass continuously many solution curves $x(t)$, distinguished one from another by the values assigned to the initial velocity $\dot{x}(0)$. Write

$$\begin{aligned}\dot{x} &\equiv y\\ \dot{y} &= -\tfrac{1}{m}V'(x)\end{aligned}$$

The original *single* 2^{nd}-*order* equation has been displayed as a *pair of* 1^{st}-*order* equations; the advantages thus gained have to do with the circumstances that

- it is often easier to solve 1^{st}-order systems (even coupled systems of them);
- it is easier to comprehend the geometry of the solution space, since through each initial point $\{x(0), y(0)\}$ passes but a *single* solution $\{x(t), y(t)\}$.

It is just such a procedure (cunningly implemented) which sends

$$\text{Lagrangian formalism} \longrightarrow \text{Hamiltonian formalism}$$

Processes of the type

$$\begin{array}{lcl} G(\dddot{x}, \ddot{x}, \dot{x}, x) = 0 & & \\ \quad\downarrow & & \\ \quad\dddot{x} = g(\ddot{x}, \dot{x}, x) & \longrightarrow & \begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = g(z, y, x) \end{cases} \end{array}$$

are called "reduction to canonical form," and are the frequently-encountered first step in work involving differential equations, whether ordinary or partial.

Look back again, in this light, to the Klein-Gordon equations

$$\big(\Box + \varkappa^2\big)\psi = 0 \quad \text{and its complex conjugate} \tag{20}$$

Define

$$\varkappa\,\psi_\mu \equiv \partial_\mu \psi \tag{33.1}$$

where the $\varkappa$-factor has been introduced in the presumption that $\varkappa \neq 0$, and in order to insure dimensional homogeneity: $[\psi_\mu] = [\psi]$. Then (20) becomes

$$\partial_\mu \psi^\mu + \varkappa\,\psi = 0 \tag{33.2}$$

[27] See, in this connection, the discussion on p. 91 in RELATIVISTIC CLASSICAL FIELDS (1973).

Equations (33) can be notated

$$\left.\begin{aligned} \varkappa\,\psi + \partial_\mu \psi^\mu &= 0 \\ \partial_\mu \psi - \varkappa\, g_{\mu\nu}\psi^\nu &= 0 \end{aligned}\right\} \tag{34}$$

or again

$$\begin{pmatrix} \varkappa & \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ \partial_0 & -\varkappa & 0 & 0 & 0 \\ \partial_1 & 0 & +\varkappa & 0 & 0 \\ \partial_2 & 0 & 0 & +\varkappa & 0 \\ \partial_3 & 0 & 0 & 0 & +\varkappa \end{pmatrix} \begin{pmatrix} \psi \\ \psi^0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{35}$$

We began with a single partial differential equation of 2^{nd}-order, and ended up with a quintet of partial differential equations of 1^{st}-order.

Changing the sign of entries on the second (the 0^{th}) row, we find ourselves in position to write

$$\big(\boldsymbol{G}^\mu \partial_\mu + \varkappa \boldsymbol{I}\big)\Psi = 0 \tag{36}$$

where Ψ is the obvious 5-element column vector, and where the matrices $\boldsymbol{G}^\mu$ are defined

$$\left.\begin{aligned} \boldsymbol{G}^0 &\equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad & \boldsymbol{G}^1 &\equiv \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \boldsymbol{G}^2 &\equiv \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad & \boldsymbol{G}^3 &\equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}\right\} \tag{37}$$

If we multiply i into (36) and define

$$\boldsymbol{\Gamma}^\mu \equiv i\,\boldsymbol{G}^\mu \tag{38}$$

then we obtain an equation

$$\big(\boldsymbol{\Gamma}^\mu \partial_\mu + i\varkappa \boldsymbol{I}\big)\Psi = 0 \tag{39}$$

which anticipates the structural design of the celebrated Dirac equation.

We will look in a moment to some of the important (and very beautiful) algebraic ramifications of the preceding formal development. But first we consider this question: Under what conditions can multi-component equations of the form

$$\big(\boldsymbol{\Gamma}^\mu \partial_\mu + i\varkappa \boldsymbol{I}\big)\Psi = 0 \quad : \quad \text{no special properties now ascribed to } \boldsymbol{\Gamma}^\mu$$

be brought within the compass of Lagrangian field theory? We expect the Lagrangian—if it exists—to be bilinear in starred and unstarred field variables, and look therefore to systems of the general type

$$\begin{aligned}\mathcal{L} &= \Psi^{*a}_{\mu}P^{\mu}_{ab}\Psi^{b} + \Psi^{a}Q^{\mu}_{ab}\Psi^{b}_{\mu} + \varkappa\,\Psi^{a}G_{ab}\Psi^{b}\\ &= \Psi^{\dagger}_{\mu}\boldsymbol{P}^{\mu}\Psi + \Psi^{\dagger}\boldsymbol{Q}^{\mu}\Psi_{\mu} + \varkappa\,\Psi^{\dagger}\boldsymbol{G}\Psi\end{aligned}$$

where the dagger † signifies hermitian conjugation, where I have once again found it convenient to omit the commas which signify partial differentiation (writing Ψ_{μ} in place of $\Psi_{,\mu}$), and where the introduction of $\varkappa$ entails no loss of generality, but simplifies the endgame. The reality of $\mathcal{L}$ entails that

$$\boldsymbol{G}^{\dagger} = \boldsymbol{G} \quad\text{and}\quad (\boldsymbol{G}^{\mu})^{\dagger} = \boldsymbol{Q}$$

Resolving $\boldsymbol{P}^{\mu}$ into its hermitian and antihermitian parts, we have

$$\begin{aligned}\boldsymbol{P}^{\mu} &= \boldsymbol{R}^{\mu} + i\tfrac{1}{2}\boldsymbol{S}^{\mu}\\ \boldsymbol{Q}^{\mu} &= \boldsymbol{R}^{\mu} - i\tfrac{1}{2}\boldsymbol{S}^{\mu}\end{aligned}$$

where $\boldsymbol{R}^{\mu}$ and $\boldsymbol{S}^{\mu}$ are both hermitian, and where the purpose of the $\frac{1}{2}$ will become clear almost immediately. The Lagrangian has now become

$$\mathcal{L} = \underbrace{\big\{\Psi^{\dagger}_{\mu}\boldsymbol{R}^{\mu}\Psi + \Psi^{\dagger}\boldsymbol{R}^{\mu}\Psi_{\mu}\big\}}_{\text{gauge term: abandon}} + i\tfrac{1}{2}\big\{\Psi^{\dagger}_{\mu}\boldsymbol{S}^{\mu}\Psi - \Psi^{\dagger}\boldsymbol{S}^{\mu}\Psi_{\mu}\big\} + \varkappa\,\Psi^{\dagger}\boldsymbol{G}\Psi$$

The resulting field equations read

$$\boldsymbol{S}^{\mu}\Psi_{\mu} + i\,\boldsymbol{G}\Psi = 0 \quad\text{and hermitian conjugate}$$

We are brought to the conclusion that $\big(\boldsymbol{\Gamma}^{\mu}\partial_{\mu} + i\varkappa\boldsymbol{I}\big)\Psi = 0$ can be obtained from a Lagrangian if and only if there exists a non-singular hermitian matrix $\boldsymbol{G}$ such that $\boldsymbol{S}^{\mu} \equiv \boldsymbol{G}\boldsymbol{\Gamma}^{\mu}$ is hermitian. The Lagrangian can, in that case, be described

$$\mathcal{L} = i\tfrac{1}{2}\big\{\Psi^{\dagger}_{\mu}\boldsymbol{G}\boldsymbol{\Gamma}^{\mu}\Psi - \Psi^{\dagger}\boldsymbol{G}\boldsymbol{\Gamma}^{\mu}\Psi_{\mu}\big\} + \varkappa\,\Psi^{\dagger}\boldsymbol{G}\Psi$$

and if we agree to write

$$\tilde{\Psi} \equiv \Psi^{\dagger}\boldsymbol{G} \tag{40}$$

becomes

$$\mathcal{L} = i\tfrac{1}{2}\big\{\tilde{\Psi}_{\mu}\boldsymbol{\Gamma}^{\mu}\Psi - \tilde{\Psi}\boldsymbol{\Gamma}^{\mu}\Psi_{\mu}\big\} + \varkappa\,\tilde{\Psi}\Psi \tag{41}$$

At (39) we had

$$\boldsymbol{\Gamma}^{0} \equiv \begin{pmatrix} 0 & i & 0 & 0 & 0\\ -i & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} \qquad \boldsymbol{\Gamma}^{1} \equiv \begin{pmatrix} 0 & 0 & i & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ +i & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$$\boldsymbol{\Gamma}^{2} \equiv \begin{pmatrix} 0 & 0 & 0 & i & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ +i & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} \qquad \boldsymbol{\Gamma}^{3} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & i\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ +i & 0 & 0 & 0 & 0\end{pmatrix}$$

and see now by inspection that a "hermitianizer" which works is

$$\boldsymbol{G} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{42}$$

I turn now to algebraic aspects of Klein-Gordon theory. Let (36) be notated

$$\boldsymbol{M}(\partial)\Psi = 0 \tag{43}$$

where

$$\boldsymbol{M}(\partial) \equiv \boldsymbol{G}^\mu \partial_\mu + \varkappa \boldsymbol{I} = \begin{pmatrix} \varkappa & \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ -\partial_0 & \varkappa & 0 & 0 & 0 \\ \partial_1 & 0 & \varkappa & 0 & 0 \\ \partial_2 & 0 & 0 & \varkappa & 0 \\ \partial_3 & 0 & 0 & 0 & \varkappa \end{pmatrix} \tag{44}$$

and where my notation is intended to emphasize that $\boldsymbol{M}(\partial)$ has the character of a matrix-valued differential operator. Calculation gives

$$\det \boldsymbol{M}(\partial) = \varkappa^3 \big(\Box + \varkappa^2\big)$$

while matrix theory supplies the general proposition that

$$\boldsymbol{M}^{-1} = \frac{(\text{matrix of cofactors})^{\mathsf{T}}}{\det \boldsymbol{M}}$$

We conclude in the case at hand that

$$\boldsymbol{W}(\partial) \equiv \frac{(\text{matrix of cofactors})^{\mathsf{T}}}{\varkappa^3}$$

has the property that

$$\boldsymbol{W}(\partial) \cdot \big(\boldsymbol{G}^\mu \partial_\mu + \varkappa \boldsymbol{I}\big) = \big(\Box + \varkappa^2\big)\boldsymbol{I} \tag{45}$$

From (45)—i.e., from the mere *existence* of such a $\boldsymbol{W}(\partial)$—it follows that if Ψ satisfies (43) then the components of Ψ *individually* satisfy the Klein-Gordon equation: $\big(\Box + \varkappa^2\big)\psi^a = 0$.[28] In the context immediately at hand this is not

---

[28] Recall from electrodynamics the fairly tricky little argument which leads from the free-field equations

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{and} \quad \partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$$

to the conclusion that

$$\Box F^{\mu\nu} = 0 \quad : \quad \text{all } \mu \text{ and } \nu$$

startling news,[29] but it becomes powerfully informative when (as below) one undertakes to enlarge upon the present context.

One can proceed computationally from (45) to an explicit description of $\boldsymbol{W}(\partial)$—*Mathematica* 4.0, running on my PowerMac G3 took .0166667 seconds to do the job; by hand it takes a bit longer—but the result is so complicated, and (seemingly) so obscurely patterned, that it would serve no useful purpose to write it out; to do so would, however, make obvious this fact:

> The elements of $\boldsymbol{W}(\partial)$ present constants, ∂ and $\partial\partial$ operators, but no derivative operators of higher than second order. (46)

Let us now look *in general* to coupled first-order field equations of the design

$$\big(\boldsymbol{G}^\mu\partial_\mu + \varkappa\boldsymbol{I}\,\big)\Psi = 0 \tag{36 ≡ 47}$$

—abandoning all special assumptions concerning

- the number of the field components ψ^a
- structural particulars of the matrices $\boldsymbol{G}^\mu$

but insisting that it remain possible to write (45); we look, in other words, to those instances of (47) for which it can be argued that if Ψ satisfies (47) then the individual components of Ψ satisfy the

$$\text{“Klein-Gordon condition”} \quad : \quad \big(\square + \varkappa^2\big)\psi^a = 0 \quad (\text{all } a) \tag{48}$$

To that end: write

$$\boldsymbol{W}(\partial) = \boldsymbol{A} + \boldsymbol{A}^\mu\partial_\mu + \boldsymbol{A}^{\mu\nu}\partial_\mu\partial_\nu + \cdots + \boldsymbol{A}^{\mu\nu\ldots\sigma}\partial_\mu\partial_\nu\cdots\partial_\sigma \tag{49}$$

Then[30]

$$\begin{aligned}\boldsymbol{W}(\partial)\cdot\big(\boldsymbol{G}^\mu\partial_\mu + \varkappa\boldsymbol{I}\,\big) &= \varkappa\boldsymbol{A} + \big(\varkappa\boldsymbol{A}^\mu + \boldsymbol{A}\boldsymbol{G}^\mu\big)\partial_\mu \\ &\quad + \big(2\varkappa\boldsymbol{A}^{\mu\nu} + [\boldsymbol{A}^\mu\boldsymbol{G}^\nu + \boldsymbol{A}^\nu\boldsymbol{G}^\mu]\big)\partial_\mu\partial_\nu \\ &\quad + \big(3\varkappa\boldsymbol{A}^{\mu\nu\rho} + [\boldsymbol{A}^{\mu\nu}\boldsymbol{G}^\rho + \boldsymbol{A}^{\nu\rho}\boldsymbol{G}^\mu + \boldsymbol{A}^{\rho\mu}\boldsymbol{G}^\nu]\big)\partial_\mu\partial_\nu\partial_\rho \\ &\quad + \cdots \\ &= \varkappa^2\boldsymbol{I} + 2g^{\mu\nu}\boldsymbol{I}\partial_\mu\partial_\nu \quad : \quad \text{REQUIRED}\end{aligned}$$

[29] Recall from (35) how the ψ^a were defined, and observe that if ψ satisfies the K-G equation then certainly the functions $\partial^\mu\psi$ do.

[30] We intend term-wise comparison of the coefficients of $\partial_\mu\partial_\nu\cdots\partial_\sigma$, and therefore (to avoid implications of the circumstance that such expressions, by their total symmetry, wear multiple aliases) understand it to be the case that

$$\sum_{\text{Greek indices}} \text{ are subject to the constraint that } \mu \leqslant \nu \leqslant \cdots \leqslant \sigma$$

Additionally, we recognize the symmetry of $g^{\mu\nu}$ and the total symmetry of $\boldsymbol{A}^{\mu\nu\ldots\sigma}$. This accounts for the intrusive integers and some otherwise inexplicable features of the following equations.

gives

$$\begin{aligned}
\varkappa \boldsymbol{A} &= \varkappa^2 \boldsymbol{I}\\
\varkappa \boldsymbol{A}^\mu + \boldsymbol{A}\boldsymbol{G}^\mu &= \boldsymbol{0}\\
2\varkappa \boldsymbol{A}^{\mu\nu} + [\boldsymbol{A}^\mu\boldsymbol{G}^\nu + \boldsymbol{A}^\nu\boldsymbol{G}^\mu] &= 2g^{\mu\nu}\boldsymbol{I}\\
3\varkappa \boldsymbol{A}^{\mu\nu\rho} + [\boldsymbol{A}^{\mu\nu}\boldsymbol{G}^\rho + \boldsymbol{A}^{\nu\rho}\boldsymbol{G}^\mu + \boldsymbol{A}^{\rho\mu}\boldsymbol{G}^\nu] &= \boldsymbol{0}\\
&\vdots
\end{aligned}$$

which can be solved serially, to give

$$\begin{aligned}
\boldsymbol{A} &= \varkappa \boldsymbol{I} && (50.0)\\
\boldsymbol{A}^\mu &= -\boldsymbol{G}^\mu && (50.1)\\
\boldsymbol{A}^{\mu\nu} &= \tfrac{1}{\varkappa}\big\{g^{\mu\nu}\boldsymbol{I} + \tfrac{1}{2}\big[\boldsymbol{G}^\mu\boldsymbol{G}^\nu + \boldsymbol{G}^\nu\boldsymbol{G}^\mu\big]\big\} && (50.2)\\
\boldsymbol{A}^{\mu\nu\rho} &= -\tfrac{1}{3\varkappa^2}\sum_{\text{cyclic permutations}}\big\{g^{\mu\nu}\boldsymbol{G}^\rho + \tfrac{1}{2}\big[\boldsymbol{G}^\mu\boldsymbol{G}^\nu + \boldsymbol{G}^\nu\boldsymbol{G}^\mu\big]\boldsymbol{G}^\rho\big\} && (50.3)\\
&\vdots
\end{aligned}$$

It is clear from the argument that gave (50) that if all $\boldsymbol{A}$-matrices of order m vanish, then so also do all $\boldsymbol{A}$-matrices of order $n > m$. The series (49) then truncates at order $m-1$.

Suppose, for example, that the matrices $\boldsymbol{G}^\mu$ have the property that

$$\sum_{\text{cyclic permutations}}\big\{g^{\mu\nu}\boldsymbol{G}^\rho + \tfrac{1}{2}\big[\boldsymbol{G}^\mu\boldsymbol{G}^\nu + \boldsymbol{G}^\nu\boldsymbol{G}^\mu\big]\boldsymbol{G}^\rho\big\} = \boldsymbol{0} \quad : \quad \text{all } \mu,\, \nu,\, \rho \tag{51}$$

Then introduction of (50) into (49) gives

$$\boldsymbol{W}(\partial) = \tfrac{1}{\varkappa}\big(\Box + \varkappa^2\big)\boldsymbol{I} - \boldsymbol{G}^\mu\partial_\mu + \tfrac{1}{2\varkappa}\big[\boldsymbol{G}^\mu\boldsymbol{G}^\nu + \boldsymbol{G}^\nu\boldsymbol{G}^\mu\big]\partial_\mu\partial_\nu \tag{52}$$

One can with labor show that the 5×5 matrices $\boldsymbol{G}^\mu$ introduced at (37) do in fact satisfy (51). And that (52) provides a structured account of precisely the "obscurely patterned" matrix calculated by *Mathematica*!

Conditions (51) can be shown[31] to be expressible

$$\boldsymbol{\Gamma}^\mu\boldsymbol{\Gamma}^\nu\boldsymbol{\Gamma}^\rho + \boldsymbol{\Gamma}^\rho\boldsymbol{\Gamma}^\nu\boldsymbol{\Gamma}^\mu = g^{\mu\nu}\boldsymbol{\Gamma}^\rho + g^{\rho\nu}\boldsymbol{\Gamma}^\mu$$

where the $\boldsymbol{\Gamma}$-matrices are those introduced at (38). The latter relations give rise to what is called "Kemmer-Duffin algebra," which was first studied in the late 1930's.[32] In higher order one encounters algebraic structures of increasing complexity, about which progressively less is known.

---

[31] See pp. 133–136 in RELATIVISTIC CLASSICAL FIELDS (1973).

[32] R. J. Duffin, "On the characteristic matrices of covariant systems," Phys. Rev. **54**, 1114 (1938); N. Kemmer, "The particle aspect of meson theory," Proc. Roy. Soc. (London) **173A**, 91 (1939). The topic had been explored already by G. Petiau in 1936.

Simplest case: the Dirac equation. The theory of a scalar Klein-Gordon field, when expressed in canonical form, was seen to lead to a 5-component Ψ-field, and to an instance of (49) which truncates in second order. More complicated covariant field theories result when we demand truncation in third, fourth or higher order. But what happens if we insist upon truncation in second (the lowest possible) order?

To do so is to impose upon the $\boldsymbol{G}$-matrices the requirement that

$$g^{\mu\nu}\boldsymbol{I} + \tfrac{1}{2}\big[\boldsymbol{G}^\mu\boldsymbol{G}^\nu + \boldsymbol{G}^\nu\boldsymbol{G}^\mu\big] = \mathbf{0} \quad : \quad \text{all } \mu \text{ and } \nu$$

which in terms of the $\boldsymbol{\Gamma}$-matrices are those introduced at (38) becomes[33]

$$\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu = 2g^{\mu\nu}\boldsymbol{I} \quad : \quad \text{all } \mu \text{ and } \nu \tag{53.1}$$

The generic field equation (36) reads[34]

$$\big(\boldsymbol{G}^\mu\partial_\mu + \varkappa\boldsymbol{I}\big)\psi = 0$$

and introduction of (50) into (49) now gives $\boldsymbol{W}(\partial) = \varkappa\boldsymbol{I} - \boldsymbol{G}^\mu\partial_\mu$ whence

$$\big(-\boldsymbol{G}^\mu\partial_\mu + \varkappa\boldsymbol{I}\big)\big(\boldsymbol{G}^\nu\partial_\nu + \varkappa\boldsymbol{I}\big)\psi = \big(\Box + \varkappa^2\big)\psi$$

In $\boldsymbol{\gamma}$-notation the preceding equations read

$$\big(\boldsymbol{\gamma}^\mu\partial_\mu + i\varkappa\boldsymbol{I}\big)\psi = 0 \tag{53.2}$$

$$\big(\boldsymbol{\gamma}^\mu\partial_\mu - i\varkappa\boldsymbol{I}\big)\big(\boldsymbol{\gamma}^\nu\partial_\nu + i\varkappa\boldsymbol{I}\big)\psi = \big(\Box + \varkappa^2\big)\psi \tag{53.3}$$

Equation (53.2) is precisely the *Dirac equation*, put forward in (1927) by P. A. M. Dirac on the basis of quite a different set of considerations.[35] Dirac was able and content simply to pluck out of thin air a quartet of complex 4×4

---

[33] I will, in fact, write $\boldsymbol{\gamma}^\mu$ in place of $\boldsymbol{\Gamma}^\mu$—partly to underscore the fact that we have particularized a generic situation, but mainly to come into agreement with long-established notational convention.

[34] Since ψ is, in the generic case, not a preempted symbol (no scalar field is a presumed player) I will henceforth write ψ where formerly we wrote Ψ; the latter symbol will see service again when we turn to the *transformational* aspects of the theory.

[35] "The quantum theory of the electron," Proc. Roy. Soc. (London) **117A**, 610 (1928) and **118A**, 351 (1928); Chapter 11, *The Principles of Quantum Mechanics* (4th edition 1958). Dirac noted that, while for a non-relativistic free particle $E = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2)$ goes over by Schrödinger quantization into a perfectly acceptable Schrödinger equation, in relativity one has

$$g^{\mu\nu}p_\mu p_\nu = (E/c)^2 - (p_1^2 + p_2^2 + p_3^2) = (mc)^2$$

which by the same procedure yields a quantum mechanically unacceptable second time derivative. Writing $E = \pm\sqrt{(mc)^2 + p_1^2 + p_2^2 + p_3^2}$ does not help, matrices which satisfy (53.1), but soon it was shown (by Pauli) that

- the complex matrices of *least* dimension which satisfy (53.1) are 4×4;
- if 4×4 quartets $\tilde{\boldsymbol{\gamma}}^\mu$ and $\boldsymbol{\gamma}^\mu$ both satisfy (53.1) then necessarily there exists a $\boldsymbol{S}$ such that $\tilde{\boldsymbol{\gamma}}^\mu = \boldsymbol{S}\boldsymbol{\gamma}^\mu\boldsymbol{S}^{-1}$.

The latter fact ensures that we do not have "disjoint realizations" to contend with, and accounts for the fact that one encounters diverse realizations in the literature. The following "Bjorken & Drell" realization

$$\left.\begin{aligned} \boldsymbol{\gamma}^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ \boldsymbol{\gamma}^2 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{aligned}\right\} \tag{54}$$

can in terms of the Pauli matrices

$$\boldsymbol{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be notated

$$\boldsymbol{\gamma}^0 = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{I} \end{pmatrix}, \qquad \boldsymbol{\gamma}^1 = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\gamma}^2 = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{\gamma}^3 = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma}_3 \\ \boldsymbol{\sigma}_3 & \mathbf{0} \end{pmatrix}$$

and enjoy fairly wide usage.[36] Note that while the Pauli matrices are hermitian, the matrices $\boldsymbol{\gamma}^m$ $(m = 1, 2, 3)$ are antihermitian. They can, however, be

---

for to do so introduces a sign ambiguity, and fractures the energy-momentum symmetry upon which relativity insists. Dirac's idea was to achieve a symmetry-preserving first-order expression by *factoring* $g^{\mu\nu}p_\mu p_\nu$, writing

$$g^{\mu\nu}p_\mu p_\nu = (\gamma^\mu p_\mu)(\gamma^\nu p_\nu)$$

even though this entails that the γ^μ be "hypernumbers" (matrices) constrained to satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Dirac appears to have been unaware that "formal factorization of quadratic forms" is a topic which had been explored by W. K. Clifford (1845–1879) already in 1876, and that γ-algebra ("Dirac algebra") provides a special instance of a *Clifford algebra*.

[36] They are, in particular, employed by David Griffiths; see p. 216 of his *Introduction to Elementary Particles* (1987).

"hermitianized," which is to say: there exists a non-singular hermitian $\boldsymbol{G}$ with the property that $\boldsymbol{G\gamma}^\mu$ is hermitian (all μ). The $\boldsymbol{G}$ which transparently does the trick is

$$\boldsymbol{G} \equiv \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{I} \end{pmatrix} \text{ which, as it happens, is just } \boldsymbol{\gamma}^0 \tag{55}$$

We are in position now to state (see again (40/41)) that *the Dirac theory admits of Lagrangian formulation*, and that the Lagrangian in question can be written

$$\begin{aligned} \mathcal{L} &= -\hbar c\Big\{i\tfrac{1}{2}\big\{\tilde{\psi}_\mu\boldsymbol{\gamma}^\mu\psi - \tilde{\psi}\boldsymbol{\gamma}^\mu\psi_\mu\big\} + \varkappa\,\tilde{\psi}\psi\Big\} \\ &= \hbar c\left\{\frac{\tilde{\psi}_\mu\boldsymbol{\gamma}^\mu\psi - \tilde{\psi}\boldsymbol{\gamma}^\mu\psi_\mu}{2i} - \varkappa\,\tilde{\psi}\psi\right\} \end{aligned} \tag{56}$$

with $\tilde{\psi} \equiv \psi^\dagger\boldsymbol{G}$. The $\hbar c$-factor has been introduced in order to ensure that $[\mathcal{L}] = \text{energy}/(\text{length})^3$, and in the presumption that $[\tilde{\psi}\psi] = 1/(\text{length})^4$, and the minus sign is physically inconsequential/cosmetic.

Drawing upon (1–41) we find that the energy density implicit in (56) might be described

$$\begin{aligned} \mathcal{E} &= \tilde{\psi}_0\frac{\partial\mathcal{L}}{\partial\tilde{\psi}_0} + \frac{\partial\mathcal{L}}{\partial\psi_0}\psi_0 - \mathcal{L} \\ &= \hbar c\left\{\frac{\tilde{\psi}_0\boldsymbol{\gamma}^0\psi - \tilde{\psi}\boldsymbol{\gamma}^0\psi_0}{2i}\right\} - \underbrace{\mathcal{L}}_{\text{can be dropped: see below.}} \end{aligned} \tag{57}$$

but

The generic canonical Lagrangian (41) yields field equations

$$i\boldsymbol{\Gamma}^\mu\Psi_\mu = +\varkappa\Psi \quad\text{and}\quad i\tilde{\Psi}_\mu\boldsymbol{\Gamma}^\mu = -\varkappa\tilde{\Psi}$$

which when inserted back into (41) give

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\big\{-\tilde{\Psi}\varkappa\Psi - \tilde{\Psi}\varkappa\Psi\big\} + \varkappa\tilde{\Psi}\Psi \\ &= 0 \quad \text{in numerical value} \end{aligned} \tag{58}$$

This result is not as strange as it might at first appear, for it is the *functional form*—not the numerical value—of $\mathcal{L}$ which is of dynamical consequence. Besides, such a result is familiar already from mechanics: Hamilton's canonical equations

$$\begin{aligned} \dot{q} &= +\tfrac{\partial}{\partial p}H \\ \dot{p} &= -\tfrac{\partial}{\partial q}H \end{aligned}$$

can be obtained as "Lagrange equations" from the "meta-Lagrangian"

$$L(\dot{q},\dot{p},q,p) \equiv \tfrac{1}{2}\big\{\dot{q}p - q\dot{p}\big\} - H(p,q)$$

and when inserted back *into* the meta-Lagrangian give

$$L = \tfrac{1}{2}\big\{q\tfrac{\partial}{\partial q} + p\tfrac{\partial}{\partial p}\big\}H - H$$

But if H is homogeneous of degree two then $\big\{q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p}\big\}H = 2H$ (by Euler's theorem), and we have

$$L = 0 \quad \text{in numerical value}$$

And homogeneity of degree two ($H = ap^2 + bpq + cq^2$) is precisely the condition required to ensure *linearity* of the canonical equations. Notice that because $\dot{q}$ and $\dot{p}$ enter linearly into the design of L, it is not possible to construct an associated "meta-Hamiltonian." Similar remarks pertain whenever derivatives enter linearly into the design of a Lagrangian (or Lagrange density)...as in canonical theories they invariably and characteristically do.

Noether's theorem supplies this satisfying account of the energy of the meta-Lagrangian system just mentioned:

$$\begin{aligned} E &= \big\{\dot{q}\tfrac{\partial}{\partial \dot{q}} + \dot{p}\tfrac{\partial}{\partial \dot{p}}\big\}L - L \\ &= \tfrac{1}{2}\big\{\dot{q}p - \dot{p}q\big\} - \Big\{\tfrac{1}{2}\big\{\dot{q}p - q\dot{p}\big\} - H(p,q)\Big\} \\ &= H(p,q) \end{aligned}$$

The (conserved) total energy present in a free Dirac field can be described

$$\begin{aligned} E &= \iiint \mathcal{E}\, dx^1 dx^2 dx^3 \\ &= \tfrac{1}{2}\hbar c \iiint i\Big\{\tilde{\psi}\boldsymbol{\gamma}^0\psi_0 - \tilde{\psi}_0\boldsymbol{\gamma}^0\psi\Big\}\, dx^1 dx^2 dx^3 \end{aligned}$$

But the field equations supply

$$\begin{aligned} i\boldsymbol{\gamma}^0\psi_0 &= -i\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}}\psi + \varkappa\,\psi \\ i\tilde{\psi}_0\boldsymbol{\gamma}^0 &= -i\tilde{\psi}\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - \varkappa\,\psi \end{aligned}$$

so

$$\begin{aligned} E &= \hbar c \iiint \Big\{- i\tfrac{1}{2}\tilde{\psi}\big[\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} - \boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}}\big]\psi + \varkappa\,\tilde{\psi}\psi\Big\}\, dx^1 dx^2 dx^3 \\ &= \hbar c \iiint \tilde{\psi}\Big\{- i\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + \varkappa\Big\}\psi\, dx^1 dx^2 dx^3 + i\tfrac{1}{2}\hbar c \iiint \overrightarrow{\boldsymbol{\nabla}}\cdot(\tilde{\psi}\boldsymbol{\gamma}\,\psi)\, dx^1 dx^2 dx^3 \end{aligned}$$

Integration by parts has yielded a second term which can (by the divergence theorem) be expressed $\iint(\tilde{\psi}\boldsymbol{\gamma}\,\psi)\cdot d\boldsymbol{\sigma}$ and abandoned. The resulting expression

$$E = \hbar c \iiint \tilde{\psi}\Big\{- i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + \varkappa\Big\}\psi\, dx^1 dx^2 dx^3 \tag{59}$$

is assigned the role of a "Hamiltonian" in some approaches[37] to *quantization* of the classical Dirac field.

The argument which gave (59) is not special to the Dirac field; it pertains to *all* canonically formulated Lagrangian field theories. It pertains in particular to the 5-component canonical formulation of the theory of a scalar Klein-Gordon field, and could be used to cast interesting new light upon (16.1). I shall, however, not pursue the detailed implications of this remark.

Dirac's theory, when looked upon as relativistic quantum mechanics of a particle ("quantum theory of the electron" in Dirac's phrase) proved to be a theory the explanatory power of which is surpassed only by its elegance. But my subject is classical field theory, so I must be content to leave all of that to Dirac's Chapters 11 & 12—to the advanced quantum textbooks.

Lorentz transform properties of multi-component fields. Revert again to the generic case (41):

$$\mathcal{L} = i\tfrac{1}{2}\big\{\tilde{\psi}_\mu \boldsymbol{\Gamma}^\mu \psi - \tilde{\psi}\boldsymbol{\Gamma}^\mu \psi_\mu\big\} + \varkappa\,\tilde{\psi}\,\psi$$

with $\tilde{\psi} \equiv \psi^\dagger \boldsymbol{G}$. More explicitly (and for present purposes more conveniently)

$$\mathcal{L} = i\tfrac{1}{2}\big\{\psi^\dagger_\mu \boldsymbol{G}\boldsymbol{\Gamma}^\mu \psi - \psi^\dagger \boldsymbol{G}\boldsymbol{\Gamma}^\mu \psi_\mu\big\} + \varkappa\,\psi^\dagger \boldsymbol{G}\psi \tag{60}$$

Lorentz covariance of the resulting field equations requires that each of the terms which enter into the construction of $\mathcal{L}$ be Lorentz invariant. Enlarging upon our experience with tensor fields, we postulate it to be the case that

$$\Lambda: \quad x \rightarrow X = \boldsymbol{\Lambda} x \tag{61.0}$$

causes the components of ψ to fold linearly among themselves:

$$\Lambda: \quad \psi^a(x) \rightarrow \Psi^a(X) = U^a{}_b(\Lambda)\psi^b(x(X)) \tag{61.1}$$

which we abbreviate

$$\Lambda: \quad \psi \rightarrow \Psi = \boldsymbol{U}(\Lambda)\psi$$

Specific instances of such field-transformation laws have been encountered already at (5/6/7). It follows from (61.1) that first partials of the multi-component field necessarily transform

$$\begin{aligned} \Lambda: \quad \psi^a_{,\mu}(x) \rightarrow \Psi^a_{,\mu}(X) &= U^a{}_b(\Lambda)\tfrac{\partial x^\nu}{\partial X^\mu}\psi^b_{,\nu}(x(X)) \\ &= U^a{}_b(\Lambda)\,\Lambda^\nu{}_\mu \psi^b_{,\nu}(x(X)) \end{aligned} \tag{61.2}$$

which we abbreviate

$$\Lambda: \quad \psi_\mu \rightarrow \Psi_\mu = \boldsymbol{U}(\Lambda)\,\Lambda^\nu{}_\mu \psi_\nu$$

[37] Recall (1–81) and see Schweber,[7] p. 220.

Looking back now to (60): we have

$$\begin{aligned}\mathcal{L} &= i\tfrac{1}{2}\big\{\Psi^\dagger_\mu \boldsymbol{G\Gamma}^\mu\Psi - \Psi^\dagger\boldsymbol{G\Gamma}^\mu\Psi_\mu\big\} + \varkappa\,\Psi^\dagger\boldsymbol{G}\Psi\\ &= i\tfrac{1}{2}\big\{\psi^\dagger_\nu\Lambda^\nu{}_\mu\boldsymbol{U}^\dagger\boldsymbol{G\Gamma}^\mu\boldsymbol{U}\psi - \psi^\dagger\boldsymbol{U}^\dagger\boldsymbol{G\Gamma}^\mu\boldsymbol{U}\,\Lambda^\nu{}_\mu\psi_\nu\big\} + \varkappa\,\psi^\dagger\boldsymbol{U}^\dagger\boldsymbol{GU}\psi\end{aligned}$$

and insist upon

$$= i\tfrac{1}{2}\big\{\psi^\dagger_\nu\boldsymbol{G\Gamma}^\nu\psi - \psi^\dagger\boldsymbol{G\Gamma}^\nu\psi_\nu\big\} + \varkappa\,\psi^\dagger\boldsymbol{G}\psi$$

We are led thus to the requirements

$$\begin{aligned}\boldsymbol{U}^\dagger\boldsymbol{GU} &= \boldsymbol{G}\\ \Lambda^\nu{}_\mu\boldsymbol{U}^\dagger\boldsymbol{G\Gamma}^\mu\boldsymbol{U} &= \boldsymbol{G\Gamma}^\nu\end{aligned}$$

The former requirement can be written

$$\boldsymbol{U}^{-1} = \boldsymbol{G}^{-1}\boldsymbol{U}^\dagger\boldsymbol{G} \tag{62.1}$$

which is reminiscent of the Lorentz condition $\boldsymbol{\Lambda}^{-1} = \boldsymbol{g}^{-1}\boldsymbol{\Lambda}^{\mathsf{T}}\boldsymbol{g}$ encountered at (3), and can be used to bring the second set of required relations to the form

$$\boldsymbol{U}^{-1}\boldsymbol{\Gamma}^\mu\boldsymbol{U} = \Lambda^\mu{}_\nu\boldsymbol{\Gamma}^\nu \tag{62.2}$$

How do (62) check out as they relate to the canonical formulation of a scalar Klein-Gordon field φ? In that case

$$\psi = \begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\\ \psi^4\\ \psi^5\end{pmatrix} = \begin{pmatrix}\varphi\\ \partial^0\varphi\\ \partial^1\varphi\\ \partial^2\varphi\\ \partial^3\varphi\end{pmatrix} \tag{63}$$

The top component transforms as a scalar, the bottom four as components of a contravariant 4-vector. . . so we have

$$\boldsymbol{U}(\Lambda) = \begin{pmatrix}1 & \boldsymbol{0}\\ \boldsymbol{0} & \begin{pmatrix}\Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3\\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3\\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3\\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3\end{pmatrix}\end{pmatrix} \tag{64}$$

The matrix is in this case real so $\boldsymbol{U}^\dagger = \boldsymbol{U}^{\mathsf{T}}$, and (62.1) can be notated

$$\begin{pmatrix}1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Lambda}^{\mathsf{T}}\end{pmatrix}\boldsymbol{G}\begin{pmatrix}1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Lambda}\end{pmatrix} = \boldsymbol{G} \tag{65}$$

But we found at (42) that the hermitianizer can in that case be described

$$\boldsymbol{G} = \begin{pmatrix}1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{g}\end{pmatrix} \tag{66}$$

so (62.1) is an immediate consequence of (3). To establish (62.2)—and at the same time to avoid the complexities latent in $\boldsymbol{U}^{-1}$—it is sufficient to show that for all 4-vectors a_μ

$$a_\mu \boldsymbol{\Gamma}^\mu \boldsymbol{U} = a_\mu \Lambda^\mu{}_\nu \boldsymbol{U}\boldsymbol{\Gamma}^\nu \tag{67}$$

which is an inelegantly straightforward computational assignment; recalling from (37/38) the definitions of the $\boldsymbol{\Gamma}$-matrices, we look to the left side of (67) and compute

$$\begin{pmatrix} 0 & a_0 & a_1 & a_2 & a_3 \\ -a_0 & 0 & 0 & 0 & 0 \\ +a_1 & 0 & 0 & 0 & 0 \\ +a_2 & 0 & 0 & 0 & 0 \\ +a_3 & 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{U} = \begin{pmatrix} 0 & a_\mu\Lambda^\mu{}_0 & a_\mu\Lambda^\mu{}_1 & a_\mu\Lambda^\mu{}_2 & a_\mu\Lambda^\mu{}_3 \\ -a^0 & 0 & 0 & 0 & 0 \\ -a^1 & 0 & 0 & 0 & 0 \\ -a^2 & 0 & 0 & 0 & 0 \\ -a^3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

while the expression on the right side of (67) yields

$$a_\mu\Lambda^\mu{}_0 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\Lambda^0{}_0 & 0 & 0 & 0 & 0 \\ -\Lambda^1{}_0 & 0 & 0 & 0 & 0 \\ -\Lambda^2{}_0 & 0 & 0 & 0 & 0 \\ -\Lambda^3{}_0 & 0 & 0 & 0 & 0 \end{pmatrix} + a_\mu\Lambda^\mu{}_1 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ +\Lambda^0{}_1 & 0 & 0 & 0 & 0 \\ +\Lambda^1{}_1 & 0 & 0 & 0 & 0 \\ +\Lambda^2{}_1 & 0 & 0 & 0 & 0 \\ +\Lambda^3{}_1 & 0 & 0 & 0 & 0 \end{pmatrix} + \cdots$$

The top row has precisely the desired structure, and all the 0's are correctly placed. Looking finally to the elements of the first column, the design of the Lorentz metric entails

$$\begin{aligned} & - a_\mu\Lambda^\mu{}_0\Lambda^\nu{}_0 + a_\mu\Lambda^\mu{}_1\Lambda^\nu{}_1 + a_\mu\Lambda^\mu{}_2\Lambda^\nu{}_2 + a_\mu\Lambda^\mu{}_3\Lambda^\nu{}_3 \\ =& - a_\mu\Lambda^{\mu 0}\Lambda^\nu{}_0 - a_\mu\Lambda^{\mu 1}\Lambda^\nu{}_1 - a_\mu\Lambda^{\mu 2}\Lambda^\nu{}_2 - a_\mu\Lambda^{\mu 3}\Lambda^\nu{}_3 \\ =& - a_\mu g^{\mu\nu} \\ =& - a^\nu \end{aligned}$$

which completes the demonstration.

Matrices $\boldsymbol{U}(\Lambda)$ which have (or by similarity transformation can be made to have) the block design

$$\begin{pmatrix}
\bullet & \bullet & \bullet & \bullet & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & & & & & & & & \\
& & & & \bullet & & & & & & & \\
& & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
& & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
& & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
& & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
& & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
& & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
& & & & & & & & & & & \ddots
\end{pmatrix}$$

(as exemplified by (64)) provide "reducible" representations of the group in question—here: the Lorentz group. Matrices which do not admit of such reduction (such as the sub-matrices found on the diagonal) are said to be "irreducible." It is with the enumeration/description of the irreducible representations that group representation theory is principally concerned.[38]

We turn now from Klein-Gordon theory to the more interesting problems presented by the Dirac theory. The Dirac field

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix}$$

came to be a 4-component complex field because (it was reported without proof) the least-dimensional representation of

$$\boldsymbol{\gamma}^\mu \boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu \boldsymbol{\gamma}^\mu = 2g^{\mu\nu}\boldsymbol{I} \quad : \quad \text{all } \mu \text{ and } \nu \tag{53.1}$$

is complex 4×4. It is tempting to suppose that the components of ψ transform

$$\psi \to \Psi = \boldsymbol{U}(\Lambda)\psi \tag{68}$$

as components of a 4-vector, but this cannot be the case: $\boldsymbol{U}$ was constrained at (62.1) to satisfy

$$\boldsymbol{U}^{-1} = \boldsymbol{G}^{-1}\boldsymbol{U}^\dagger \boldsymbol{G}$$

and (see again (55))

$$\boldsymbol{G} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{is non-Lorentzian } (\neq \boldsymbol{g})$$

Evidently the Dirac field ψ transforms, according to (68), as a new kind of object—a "4-spinor," and the "4-ness" of the matter has nothing (or at least nothing obvious) to do with the 4-dimensionality of spacetime.

Complex representations of the Lorentz group are recommended to our attention by the simplest of considerations: set up the association

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \longleftrightarrow \quad \boldsymbol{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \quad : \ \text{hermitian}$$

[38] See (for example) F. D. Murnaghan, *The Theory of Group Representations* (1938), Chapter 2.

and notice that

$$\det \boldsymbol{x} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = g_{\mu\nu}x^\mu x^\nu = (x,x)$$

is invariant under

$$\boldsymbol{x} \to \boldsymbol{X} = \boldsymbol{U}^{-1}\boldsymbol{x}\,\boldsymbol{U} \;\;:\;\; \boldsymbol{U} \text{ is complex } 2\times 2$$

The complex matrix $\boldsymbol{U}$ contains eight assignable constants, but if we impose the "unimodularity" condition $\det\boldsymbol{U} = 1 + i0$ that number is reduced to six. We may expect to set up an association between elements $\boldsymbol{\Lambda}$ of the 6-parameter Lorentz group and elements $\boldsymbol{S}$ of the 6-parameter group $SU(2)$ of unimodular complex 2×2 matrices (all of which are, as it happens, automatically unitary):

$$\boldsymbol{\Lambda} \quad\longleftrightarrow\quad \boldsymbol{S}(\Lambda) \;\;:\;\; \text{element of } SU(2)$$

Then

$$\boldsymbol{x} \to \boldsymbol{X} = \boldsymbol{S}^{-1}\boldsymbol{x}\,\boldsymbol{S} \quad \text{provides a representation of} \quad x \to X = \boldsymbol{\Lambda}x \tag{69.1}$$

Two-component complex objects $\psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}$ which participate by transforming

$$\psi \to \Psi = \boldsymbol{S}\,\psi \tag{69.2}$$

are said to transform as "2-spinors." In it important to notice that the association $\boldsymbol{\Lambda} \leftrightarrow \boldsymbol{S}(\Lambda)$ cannot be unique (is "biunique"), for $\boldsymbol{S}(\Lambda)$ enters quadratically into the equation which describes $\boldsymbol{x} \to \boldsymbol{X}$: the matrices $\pm\boldsymbol{S}$ both achieve the same effect, though the same cannot be said of (69.2).

At (68) we are presented with occasion to develop a grander variant of that same general kind of mathematics—a variant in which the design of $\boldsymbol{U}(\Lambda)$ is controlled by the statements

$$\boldsymbol{U}^{-1} = \boldsymbol{G}^{-1}\boldsymbol{U}^\dagger\boldsymbol{G} \tag{70.1}$$

$$\boldsymbol{U}^{-1}\boldsymbol{\gamma}^\mu\boldsymbol{U} = \Lambda^\mu{}_\nu\boldsymbol{\gamma}^\nu \tag{70.2}$$

Both equations are invariant under $\boldsymbol{U} \to -\boldsymbol{U}$, so we expect the association $\boldsymbol{\Lambda} \leftrightarrow \boldsymbol{U}(\Lambda)$ to be once again biunique. Note also that the metric structure $\boldsymbol{g}$ of spacetime enters (70.2) once through the relations $\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu = 2g^{\mu\nu}\boldsymbol{I}$ that control the design of the $\boldsymbol{\gamma}$-matrices, and once through the equation $\boldsymbol{\Lambda}^{-1} = \boldsymbol{g}^{-1}\boldsymbol{\Lambda}^{\mathsf{T}}\boldsymbol{g}$ that controls the design of the numbers $\Lambda^\mu{}_\nu$. If the $\boldsymbol{G}$-factors were absent from (70.1) then that equation would assert the *unitarity* of $\boldsymbol{U}$; noting that if we define

$$(\varphi,\psi) \equiv \varphi^\dagger\boldsymbol{G}\,\psi = \tilde{\varphi}\,\psi$$

and write $\psi \to \Psi = \boldsymbol{U}\psi$ then (70.1), written $\boldsymbol{G} = \boldsymbol{U}^\dagger\boldsymbol{G}\boldsymbol{U}$, entails

$$(\varphi,\psi) = (\Phi,\Psi)$$

We shall, on this account, say of matrices $\boldsymbol{U}$ with property (70.1) that they are "$\boldsymbol{G}$-unitary." We recover ordinary unitarity by sending $\boldsymbol{G} \to \boldsymbol{I}$. By an easy argument, $\boldsymbol{U} = e^{\boldsymbol{B}}$ will be $\boldsymbol{G}$-unitary if and only if $\boldsymbol{B} \equiv \|B^a{}_b\|$ is

$$\text{"}\boldsymbol{G}\text{-antihermitian"} \quad : \quad \boldsymbol{G}^{-1}\boldsymbol{B}^\dagger \boldsymbol{G} = -\boldsymbol{B}$$

Which (by the hermiticity of $\boldsymbol{G} = \|G_{ab}\|$) entails that $\boldsymbol{GB}$ be antihermitian in the ordinary sense: $(\boldsymbol{GB})^\dagger = -\boldsymbol{GB}$.

A similar (but more familiar) argument establishes that $\boldsymbol{\Lambda} = e^{\boldsymbol{A}}$ will be a Lorentz matrix (i.e., that it will satisfy $\boldsymbol{g} = \boldsymbol{\Lambda}^{\mathsf{T}}\boldsymbol{g}\,\boldsymbol{\Lambda}$) if and only if $\boldsymbol{A} \equiv \|A^\mu{}_\nu\|$ is

$$\text{"}\boldsymbol{g}\text{-antisymmetric"} \quad : \quad \boldsymbol{g}^{-1}\boldsymbol{A}^{\mathsf{T}}\,\boldsymbol{g} = -\boldsymbol{A}$$

Which (by the real symmetry of $\boldsymbol{g} = \|g_{\mu\nu}\|$) entails that $\boldsymbol{gA}$ be antisymmetric in the ordinary sense: $(\boldsymbol{gA})^{\mathsf{T}} = -\boldsymbol{gA}$.

Now write

$$\begin{aligned} \boldsymbol{U} &= \boldsymbol{I} + \boldsymbol{B} + \tfrac{1}{2}\boldsymbol{B}^2 + \cdots \\ \boldsymbol{U}^{-1} &= \boldsymbol{I} - \boldsymbol{B} + \tfrac{1}{2}\boldsymbol{B}^2 - \cdots \\ \boldsymbol{\Lambda} &= \boldsymbol{I} + \boldsymbol{A} + \tfrac{1}{2}\boldsymbol{A}^2 + \cdots \end{aligned}$$

stick those series into (70.2), abandon all but leading order terms, and obtain

$$\boldsymbol{\gamma}^\mu \boldsymbol{B} - \boldsymbol{B}\,\boldsymbol{\gamma}^\mu = A^\mu{}_\nu \boldsymbol{\gamma}^\nu \tag{71}$$

One could, by further cultivation of the algebra, deduce[39]—alternatively: one can, by direct computation, simply confirm—that the solution of (71) can be described

$$\boldsymbol{B} = \tfrac{1}{8}A^{\alpha\beta}(\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_\beta - \boldsymbol{\gamma}_\beta\boldsymbol{\gamma}_\alpha) \tag{72}$$

Little of the compact elegance of this result (the neat structure of which might almost have been guessed) survives when it is written out explicitly

$$\boldsymbol{B} = \tfrac{1}{2}\begin{pmatrix} 0 & A_{13} & -A_{03} & -A_{01} \\ -A_{13} & 0 & -A_{01} & A_{03} \\ -A_{03} & -A_{01} & 0 & A_{13} \\ -A_{01} & A_{03} & -A_{13} & 0 \end{pmatrix} + i\tfrac{1}{2}\begin{pmatrix} -A_{12} & -A_{23} & 0 & A_{02} \\ -A_{23} & A_{12} & -A_{02} & 0 \\ 0 & A_{02} & -A_{12} & -A_{23} \\ -A_{02} & 0 & -A_{23} & A_{12} \end{pmatrix}$$

but the explicit description is not without it uses.[40] For example: Suppose $\boldsymbol{A}$ generates *rotation about the x^3-axis*. We set

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \vartheta & 0 \\ 0 & -\vartheta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

---

[39] See, for example, Schweber,[7] p. 77.

[40] Note that I have used $A_{\mu\nu} = -A_{\nu\mu}$ to arrange that only the independently specifiable parameters $\{A_{01}, A_{02}, A_{03}, A_{12}, A_{13}, A_{23}\}$ appear in the matrix. It is gratifying to observe that

$$\boldsymbol{GB} = (\text{real antisymmetric}) + i(\text{real symmetric})$$

is in fact antihermitian (as required). Writing $\boldsymbol{B}$ and the explicit descriptions and (ask *Mathematica* to compute $\sum_0^6 \frac{1}{n!}$`MatrixPower[A,n]//MatrixForm`) are brought to the (anticipated) conclusion that

$$\mathbf{\Lambda} = e^{\boldsymbol{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta & 0 \\ 0 & -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The generator of the associated spin transformation is

$$\boldsymbol{B} = i\tfrac{1}{2}\begin{pmatrix} -\vartheta & 0 & 0 & 0 \\ 0 & +\vartheta & 0 & 0 \\ 0 & 0 & -\vartheta & 0 \\ 0 & 0 & 0 & +\vartheta \end{pmatrix}$$

which gives

$$\boldsymbol{U} = e^{\boldsymbol{B}} = \begin{pmatrix} e^{-i\frac{1}{2}\vartheta} & 0 & 0 & 0 \\ 0 & e^{+i\frac{1}{2}\vartheta} & 0 & 0 \\ 0 & 0 & e^{-i\frac{1}{2}\vartheta} & 0 \\ 0 & 0 & 0 & e^{+i\frac{1}{2}\vartheta} \end{pmatrix}$$

This result exposes the striking (and deeply consequential) fact that one must rotate 720° in physical space to achieve a 360° rotation in spin space. To describe a *boost in the x^1-direction* we would set

$$\boldsymbol{A} = \begin{pmatrix} 0 & \psi & 0 & 0 \\ \psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and obtain

$$\mathbf{\Lambda} = e^{\boldsymbol{A}} = \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(continued from the preceding page) (54) of $\boldsymbol{\Gamma}^\mu$ into *Mathematica*, it takes only typing time to confirm that

$$\begin{aligned} &\boldsymbol{\Gamma}_0\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\Gamma}_0 - (\ \ A_{01}\boldsymbol{\Gamma}^1 + A_{02}\boldsymbol{\Gamma}^2 + A_{03}\boldsymbol{\Gamma}^3) \\ &= \boldsymbol{\Gamma}_0\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\Gamma}_0 - (-A_{01}\boldsymbol{\Gamma}_1 - A_{02}\boldsymbol{\Gamma}_2 - A_{03}\boldsymbol{\Gamma}_3) = \mathbf{0} \end{aligned}$$

$$\begin{aligned} &\boldsymbol{\Gamma}_1\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\Gamma}_1 - (\ \ A_{10}\boldsymbol{\Gamma}^0 + A_{12}\boldsymbol{\Gamma}^2 + A_{13}\boldsymbol{\Gamma}^3) \\ &= \boldsymbol{\Gamma}_1\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\Gamma}_1 - (-A_{01}\boldsymbol{\Gamma}_0 - A_{12}\boldsymbol{\Gamma}_2 - A_{13}\boldsymbol{\Gamma}_3) = \mathbf{0} \end{aligned}$$

$$\begin{aligned} &\boldsymbol{\Gamma}_2\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\Gamma}_2 - (\ \ A_{20}\boldsymbol{\Gamma}^0 + A_{21}\boldsymbol{\Gamma}^1 + A_{23}\boldsymbol{\Gamma}^3) \\ &= \boldsymbol{\Gamma}_2\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\Gamma}_2 - (-A_{02}\boldsymbol{\Gamma}_0 + A_{12}\boldsymbol{\Gamma}_1 - A_{23}\boldsymbol{\Gamma}_3) = \mathbf{0} \end{aligned}$$

$$\begin{aligned} &\boldsymbol{\Gamma}_3\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\Gamma}_3 - (\ \ A_{30}\boldsymbol{\Gamma}^0 + A_{31}\boldsymbol{\Gamma}^1 + A_{32}\boldsymbol{\Gamma}^2) \\ &= \boldsymbol{\Gamma}_3\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\Gamma}_3 - (-A_{03}\boldsymbol{\Gamma}_0 + A_{13}\boldsymbol{\Gamma}_1 + A_{23}\boldsymbol{\Gamma}_2) = \mathbf{0} \end{aligned}$$

which is the upshot of the assertion that (72) satisfies (71).

The generator of the associated spin transformation is

$$\boldsymbol{B} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -\psi \\ 0 & 0 & -\psi & 0 \\ 0 & -\psi & 0 & 0 \\ -\psi & 0 & 0 & 0 \end{pmatrix}$$

which gives

$$\boldsymbol{U} = e^{\boldsymbol{B}} = \begin{pmatrix} \cosh\frac{1}{2}\psi & 0 & 0 & -\sinh\frac{1}{2}\psi \\ 0 & \cosh\frac{1}{2}\psi & -\sinh\frac{1}{2}\psi & 0 \\ 0 & -\sinh\frac{1}{2}\psi & \cosh\frac{1}{2}\psi & 0 \\ -\sinh\frac{1}{2}\psi & 0 & 0 & \cosh\frac{1}{2}\psi \end{pmatrix}$$

Contact with the more familiar kinematic parameters β and $\gamma = 1/\sqrt{1-\beta^2}$ is made by means of

$$\begin{aligned} \cosh\psi &= \gamma \\ \sinh\psi &= \gamma\beta \\ \cosh\tfrac{1}{2}\psi &= \sqrt{\tfrac{1}{2}(\gamma+1)} \\ \sinh\tfrac{1}{2}\psi &= \sqrt{\tfrac{1}{2}(\gamma-1)} \end{aligned}$$

Noether meets Einstein. Noether has given us a deep-seated mechanism for associating "current vectors" $J_r^\mu(\varphi, \partial\varphi)$ with parameterized maps (one for each parameter). And for associating conservation laws $\partial_\mu J_r^\mu = 0$ with maps which refer to symmetries of the dynamical action. In specific applications the maps in question are typically contingent—recommended to our attention by features of the dynamical system in hand.

Einstein has directed our attention to a transformation group (the Lorentz group—more generally, the Poincaré group) which the Principle of Relativity asserts to be *universal*, a symmetry shared by *all* admissible dynamical systems, and (in the absence of gravitational effects) explicitly/implicitly present in all well-designed dynamical theories. The stress-energy tensor $S^\mu{}_\nu$ refers to the ν-indexed quartet of conservation laws which arise from the translational part of the Poincaré group. We look now to the design and properties of the sextet of Noetherean currents which arise from the postulated Lorentz covariance of relativistic systems.

Our first assignment is to obtain descriptions of the structure functions characteristic of the homogeneous Lorentz group. For infinitesimal $\boldsymbol{A}$ we have

$$\boldsymbol{\Lambda} = e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \cdots$$

$$\boldsymbol{A} = \begin{pmatrix} 0 & \delta\Omega^1 & \delta\Omega^2 & \delta\Omega^3 \\ \delta\Omega^1 & 0 & +\delta\omega^3 & -\delta\omega^2 \\ \delta\Omega^2 & -\delta\omega^3 & 0 & +\delta\omega^1 \\ \delta\Omega^3 & +\delta\omega^2 & -\delta\omega^1 & 0 \end{pmatrix} \equiv \boldsymbol{B}_k\delta\Omega^k + \boldsymbol{R}_k\delta\omega^k$$

where the $\{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3\}$ generate *boosts*, and $\{\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3\}$ generate spatial *rotations*. But it is, for the purposes at hand, advantageous to proceed a bit more circumspectly, writing

$$\begin{aligned}\boldsymbol{A} &= \begin{pmatrix} 0 & \delta\Omega_{01} & \delta\Omega_{02} & \delta\Omega_{03} \\ \delta\Omega_{01} & 0 & -\delta\Omega_{12} & +\delta\Omega_{31} \\ \delta\Omega_{02} & +\delta\Omega_{12} & 0 & -\delta\Omega_{23} \\ \delta\Omega_{03} & -\delta\Omega_{31} & +\delta\Omega_{23} & 0 \end{pmatrix} \\ &\equiv \boldsymbol{N}^{01}\delta\Omega_{01} + \boldsymbol{N}^{02}\delta\Omega_{02} + \boldsymbol{N}^{03}\delta\Omega_{03} + \boldsymbol{N}^{12}\delta\Omega_{12} + \boldsymbol{N}^{23}\delta\Omega_{23} + \boldsymbol{N}^{31}\delta\Omega_{31} \\ &= \tfrac{1}{2}\boldsymbol{N}^{\alpha\beta}\delta\Omega_{\alpha\beta} \end{aligned} \tag{73}$$

subject to the understanding that $\boldsymbol{N}^{\alpha\beta} = -\boldsymbol{N}^{\beta\alpha}$, $\delta\Omega_{\alpha\beta} = -\delta\Omega_{\beta\alpha}$; the generic superscript ^{r} which entered the discussion immediately prior to (1–26) has now become an antisymmetrized *pair* of indices, while the generic $\sum_r$ has become a double sum, managed by the Einstein summation convention. One verifies by inspection that

$$\boldsymbol{N}^{\alpha\beta} = \|N^{\alpha\beta\mu}{}_\nu\| = \|(g^{\mu\alpha}\delta^\beta{}_\nu - g^{\mu\beta}\delta^\alpha{}_\nu)\| \tag{74}$$

We conclude from (74) that the effect, within spacetime, of an infinitesimal Lorentz transformation can be described (see again (1–26.1))

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + \delta_\Omega x^\mu \\ \delta_\Omega x^\mu &= \tfrac{1}{2}\mathfrak{X}^{\mu\alpha\beta}(x)\delta\Omega_{\alpha\beta} \\ \mathfrak{X}^{\mu\alpha\beta}(x) &= (g^{\mu\alpha}\delta^\beta{}_\nu - g^{\mu\beta}\delta^\alpha{}_\nu)x^\nu \\ &= (g^{\mu\alpha}x^\beta - g^{\mu\beta}x^\alpha) \end{aligned} \tag{75.1}$$

But at the same time (see again (11–26.2)) the field components $\varphi^a(x)$ fold among themselves in infinitesimal *representation* of the infinitesimal Lorentz transformation; the details are case-specific (depend upon the representation selected by the system in question), but have the generic form

$$\begin{aligned} \varphi^a(x) &\longrightarrow \varphi^a(x) + \delta_\Omega\varphi^a(x) \\ \delta_\Omega\varphi^a(x) &= \tfrac{1}{2}\Phi^{a\alpha\beta}(\varphi)\delta\Omega_{\alpha\beta} \\ \Phi^{a\alpha\beta}(\varphi) &= B^a{}_b{}^{\alpha\beta}\varphi^b \end{aligned} \tag{75.2}$$

If, for example, the physical system involved only a scalar field we would omit the component-identifier (i.e., we would write φ in place of φ^a) and would have $\Phi^{\alpha\beta} = 0$. If the system presented only a single vector field we would in place of φ^a write φ^μ, and would obtain $\Phi^{\mu\alpha\beta}(\varphi) = (g^{\mu\alpha}\varphi^\beta - g^{\mu\beta}\varphi^\alpha)$. We will look later to the relatively more interesting case of the Dirac spinor field, but for the moment proceed generically—armed only with the knowledge that $\Phi^{a\alpha\beta}(\varphi)$ is, like $\mathfrak{X}^{\mu\alpha\beta}(x)$, $\alpha\beta$-antisymmetric.

Returning with the structure functions (75) to Noether's equation (1–29), we are led to the $\alpha\beta$-indexed antisymmetric array of currents

$$\begin{aligned} J^{\mu\alpha\beta} &= \frac{\partial\mathcal{L}}{\partial\varphi^a_{,\mu}}\Big\{\varPhi^{a\alpha\beta} - \varphi^a_{,\sigma}\mathcal{X}^{\sigma\alpha\beta}\Big\} + \mathcal{L}\mathcal{X}^{\mu\alpha\beta} \\ &= -\Big\{\frac{\partial\mathcal{L}}{\partial\varphi^a_{,\mu}}\varphi^a_{,\sigma} - \mathcal{L}\delta^\mu{}_\sigma\Big\}\mathcal{X}^{\sigma\alpha\beta} + \frac{\partial\mathcal{L}}{\partial\varphi^a_{,\mu}}\varPhi^{a\alpha\beta} \end{aligned}$$

Recalling from (1–34) the construction of the stress-energy tensor,[41] and drawing upon the specific information established at (75.1), we have

$$\begin{aligned} \mathcal{J}^{\mu\alpha\beta} \equiv \tfrac{1}{c}J^{\mu\alpha\beta} &= -\tfrac{1}{c}\mathcal{T}^\mu{}_\sigma(g^{\sigma\alpha}x^\beta - g^{\sigma\beta}x^\alpha) + \tfrac{1}{c}\frac{\partial\mathcal{L}}{\partial\varphi^a_{,\mu}}\varPhi^{a\alpha\beta} \\ &= \tfrac{1}{c}(x^\alpha\mathcal{T}^{\mu\beta} - x^\beta\mathcal{T}^{\mu\alpha}) + \tfrac{1}{c}\frac{\partial\mathcal{L}}{\partial\varphi^a_{,\mu}}B^a{}_b{}^{\alpha\beta}\varphi^b \\ &= \mathcal{M}^{\mu\alpha\beta} + S^{\mu\alpha\beta} \end{aligned} \tag{76}$$

where, in terminology which it will be incumbent upon us to justify,

$$\mathcal{M}^{\mu\alpha\beta} \equiv \tfrac{1}{c}(x^\alpha\mathcal{T}^{\mu\beta} - x^\beta\mathcal{T}^{\mu\alpha}) \;:\; \begin{cases}\text{defines the ``orbital angular momentum''}\\ \text{of the relativistic field system}\end{cases}$$

$$S^{\mu\alpha\beta} \equiv \tfrac{1}{c}\frac{\partial\mathcal{L}}{\partial\varphi^a_{,\mu}}B^a{}_b{}^{\alpha\beta}\varphi^b \;:\; \begin{cases}\text{defines the ``intrinsic angular momentum''}\\ \text{or ``spin'' of the system}\end{cases}$$

The $\frac{1}{c}$-factors have been introduced to render such terminology *dimensionally* tenable:

$$\begin{aligned} [\,\mathcal{J}^{\mu\alpha\beta}\,] = [\,\mathcal{M}^{\mu\alpha\beta}\,] = [\,S^{\mu\alpha\beta}\,] &= \frac{\text{(length)(energy density)}}{\text{velocity}} \\ &= \text{angular momentum density} \end{aligned}$$

The expressions $\mathcal{J}^{\mu\alpha\beta}(\varphi,\partial\varphi,x)$ (ditto $\mathcal{M}^{\mu\alpha\beta}(\varphi,\partial\varphi,x)$, $S^{\mu\alpha\beta}(\varphi)$) transform as components of an $\alpha\beta$-antisymmetric third-rank tensor. The components of $\mathcal{J}^{\mu\alpha\beta}$ (ditto $\mathcal{M}^{\mu\alpha\beta}$, $S^{\mu\alpha\beta}$) are intermixed by Lorentz transformation, but the resolution of $\mathcal{J}^{\mu\alpha\beta}(\varphi,\partial\varphi,x)$ into an "orbital" part and an "intrinsic" part ("spin") maintains its integrity (i.e., is something all inertial observers agree upon).

From the Lorentz invariance which was built into the Lagrangian (whence into the action functional) we know it to be an implication of the (Lorentz covariant) field equations that

$$\partial_\mu\mathcal{J}^{\mu\alpha\beta} = 0 \tag{77}$$

---

[41] For the purpose of this discussion (in order to release the letter S to *spin*-like assignments) we use $\mathcal{T}^{\mu\nu}$ to denote the stress-energy tensor (which many authors call the "energy-momentum tensor.")

This is a *sextet of conservation laws, which pertain with certainty to every relativistic field theory*. Drawing motivation from this result and from the additive structure of (76), we observe that

$$\begin{aligned}\partial_\mu \mathcal{M}^{\mu\alpha\beta} &= \tfrac{1}{c}\Big\{\delta^\alpha{}_\mu \mathcal{T}^{\mu\beta} + x^\alpha \partial_\mu \mathcal{T}^{\mu\beta} - \delta^\beta{}_\mu \mathcal{T}^{\mu\alpha} + x^\beta \partial_\mu \mathcal{T}^{\mu\alpha}\Big\}\\ &= \tfrac{1}{c}\Big\{\mathcal{T}^{\alpha\beta} - \mathcal{T}^{\beta\alpha}\Big\} \quad \text{by } \partial_\mu \mathcal{T}^{\mu\nu} = 0\\ &= 0 \quad \text{if and only if the stress-energy tensor } \mathcal{T}^{\mu\nu} \text{ is } \textit{symmetric}\end{aligned}$$

Evidently (77) is, in non-symmetric cases, achieved by a kind of "trade-off" between the orbital and intrinsic parts of $\mathcal{J}^{\mu\alpha\beta}$. Such "spin-orbit coupling" is, of course, commonplace in the classical mechanics of many-body systems (tops, planetary systems), as it is also in atomic physics.

Look now to the expressions

$$\begin{aligned}M^{\alpha\beta} &\equiv \iiint \mathcal{M}^{0\alpha\beta}\, dx^1 dx^2 dx^3\\ &= \tfrac{1}{c}\iiint (x^\alpha \mathcal{T}^{0\beta} - x^\beta \mathcal{T}^{0\alpha})\, dx^1 dx^2 dx^3\end{aligned}$$

which evidently refer to *first moment properties* of the expressions

$$\mathcal{T}^{0\nu} \equiv c\mathcal{P}^\nu \;\; : \;\; \begin{cases} c\mathcal{P}^0 = \mathcal{E} & : \;\; \text{energy density}\\ \mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3 & : \;\; \text{components of momentum density}\end{cases}$$

and which in symmetric cases $\mathcal{T}^{\mu\nu} = \mathcal{T}^{\nu\mu}$ will describe global *constants of the field motion* (they will in other cases remain "interesting," even though not conserved). It is an option available to each individual inertial observer to resolve those expressions into two classes:

$$\begin{aligned} K_1 &\equiv M^{01} & L_1 &\equiv M^{23}\\ K_2 &\equiv M^{02} & L_2 &\equiv M^{31}\\ K_3 &\equiv M^{03} & L_3 &\equiv M^{12}\end{aligned}$$

Looking first to the latter, we have

$$\begin{aligned} L_1 &= \iiint \big(x^2\mathcal{P}^3 - x^3\mathcal{P}^2\big)\, dx^1 dx^2 dx^3\\ L_2 &= \iiint \big(x^3\mathcal{P}^1 - x^1\mathcal{P}^3\big)\, dx^1 dx^2 dx^3\\ L_3 &= \iiint \big(x^1\mathcal{P}^2 - x^2\mathcal{P}^1\big)\, dx^1 dx^2 dx^3\end{aligned}$$

which lend a clear and natural meaning to the *orbital angular momentum density* $\boldsymbol{\mathcal{L}} \equiv \boldsymbol{x} \times \boldsymbol{\mathcal{P}}$ of a relativistic field system. An identical train of thought leads one to write

$$K_1 = c\iiint \big(t\mathcal{P}^1 - x^1\mathcal{M}\big)\,dx^1dx^2dx^3$$
$$K_2 = c\iiint \big(t\mathcal{P}^2 - x^2\mathcal{M}\big)\,dx^1dx^2dx^3$$
$$K_3 = c\iiint \big(t\mathcal{P}^3 - x^3\mathcal{M}\big)\,dx^1dx^2dx^3$$

where we have used $x^0 = ct$ and introduced

$$\mathcal{M} \equiv \mathcal{E}/c^2 \;\;:\;\text{equivalent “mass density”}$$

Write

$$\begin{aligned} M &\equiv \iiint \mathcal{M}\,dx^1dx^2dx^3 = \text{conserved “total mass” of the field system}\\ &\qquad\qquad\qquad\qquad\;\; = (\text{conserved total energy})/c^2\\ \boldsymbol{X} &\equiv \tfrac{1}{M}\iiint \boldsymbol{x}\,\mathcal{M}\,dx^1dx^2dx^3 = \text{“center of mass” of the field system}\\ \boldsymbol{P} &\equiv \iiint \boldsymbol{\mathcal{P}}\,dx^1dx^2dx^3 = \text{conserved total momentum}\end{aligned}$$

and obtain the conservation of $\boldsymbol{K} = t\boldsymbol{P} - M\boldsymbol{X}$, which can be expressed

$$\boldsymbol{X}(t) = \tfrac{1}{M}\boldsymbol{P}\,t - \boldsymbol{K}$$

We are brought thus to the conclusion that *if the stress-energy tensor* $\mathcal{T}^{\mu\nu}$ *is symmetric* (!) then

- orbital angular momentum $\boldsymbol{L} \equiv \iiint \boldsymbol{\mathcal{L}}\,dx^1dx^2dx^3$ is conserved;
- the center of mass $\boldsymbol{X}(t)$ of the field system moves uniformly/rectilinearly;
- six conservation laws $\partial_\mu S^{\mu\alpha\beta} = 0$ refer to the “conservation of spin.”

Belinfante’s fandango. The pretty results obtained in discussion subsequent to (77) are, as I have several times stressed, contingent upon the *symmetry of the stress-energy tensor*. But introduction of (76) into (77) gives

$$\mathcal{T}^{\alpha\beta} - \mathcal{T}^{\beta\alpha} + c\,\partial_\mu S^{\mu\alpha\beta} = 0 \tag{78}$$

so in point of fact we have $\mathcal{T}^{\alpha\beta} = \mathcal{T}^{\beta\alpha}$ only with respect to field components φ^a and φ^b which transform as *scalars*.[42] However...

Noetherian currents generally (and the stress-energy tensor in particular) respond non-trivially to unphysical gauge transformations:[43]

$$\mathcal{L} \longrightarrow L = \mathcal{L} + \partial_\sigma \mathcal{G}^\sigma(\varphi) \tag{79.1}$$

[42] Such fields are too simple to respond to Lorentz transformation by “folding among themselves,” and $B^a{}_b{}^{\alpha\beta} = 0$ forces $\mathcal{S}^{\mu\alpha\beta} = 0$.

[43] Here I simply adapt to the language of *relativistic* field theory (and extend) the remarks which accompanied (1–36).

induces

$$\mathcal{J}_r^\mu \longrightarrow J_r^\mu = \mathcal{J}_r^\mu + G_r^\mu(\varphi, \partial\varphi) \tag{79.2}$$
$$G_r^\mu(\varphi, \partial\varphi) = \Big\{\big[\varPhi_{ar} - \varphi_{a,\rho}\mathcal{X}_r^\rho\big]\frac{\partial}{\partial\varphi_{a,\mu}} + \mathcal{X}_r^\mu\Big\}\big(\partial_\sigma\mathcal{G}^\sigma\big)$$

so one must be circumspect when ascribing "direct physical significance" to properties of such currents. In relativistic field theory it is natural to require of $\mathcal{L} \longrightarrow L$ that it respect the "principle of Lagrangian Lorentz-invariance," which is automatic if one admits only expressions $\mathcal{G}^\sigma(\varphi)$ which transform as 4-vectors. The gauge transformation will "preserve conservation laws"

$$\partial_\mu \mathcal{J}_r^\mu = 0 \quad \Longrightarrow \quad \partial_\mu J_r^\mu = 0 \quad \text{if and only if } \partial_\mu G_r^\mu = 0 \tag{80.1}$$

and the latter condition becomes automatic in cases where G_r^ν can be described

$$G_r^\nu(\varphi, \partial\varphi) = \partial_\mu H_r^{\mu\nu}(\varphi) \quad \text{with} \quad H_r^{\mu\nu} = -H_r^{\nu\mu} \tag{80.2}$$

From

$$\begin{aligned}
\mathcal{J}_r &= \iiint_{\mathcal{R}} \mathcal{J}_r^0\, dx^1 dx^2 dx^3\\
&= \iiint_{\mathcal{R}} J_r^0\, dx^1 dx^2 dx^3 + \iiint_{\mathcal{R}} \boldsymbol{\nabla}\!\cdot\!\boldsymbol{H}_r\, dx^1 dx^2 dx^3 \quad : \quad \text{here } \boldsymbol{H}_r \equiv \begin{pmatrix} H_r^{10}\\ H_r^{20}\\ H_r^{30}\end{pmatrix}\\
&= \iiint_{\mathcal{R}} J_r^0\, dx^1 dx^2 dx^3 + \iint_{\partial\mathcal{R}} \boldsymbol{H}_r\!\cdot d\boldsymbol{\sigma}\\
&= J_r \quad \text{on the presumption that the surface integral vanishes as } \partial\mathcal{R} \to \infty
\end{aligned}$$

we see that $\partial_\mu \mathcal{J}_r^\mu = 0$ and $\partial_\mu J_r^\mu = 0$ will then refer in distinct local dialects to the global conservation of the same net things. . . to which, however, they ascribe distinct densities $\mathcal{J}_r^0 \neq J_r^0$ and correspondingly distinct fluxes $\boldsymbol{\mathcal{J}}_r \neq \boldsymbol{J}_r$.

Look now, within the framework of those remarks, to the stress-energy (which arises from the translational part of the Poincaré group; the generic $_r$ becomes now the Greek index $_\nu$). We have

$$\mathcal{T}^\mu{}_\nu \longrightarrow T^\mu{}_\nu = \mathcal{T}^\mu{}_\nu + \partial_\sigma H^{\sigma\mu}{}_\nu \tag{81}$$

If $\mathcal{L} \longrightarrow L$ preserves translational invariance then it will assuredly be the case that both $\partial_\mu \mathcal{T}^\mu{}_\nu = 0$ and $\partial_\mu T^\mu{}_\nu = 0$; in the preceding equation the latter statement is presented as a corollary of the former, attributable to the $\sigma\mu$-antisymmetry of $H^{\sigma\mu}{}_\nu$.

The idea now—original to F. J. Belinfante[44]—is to use gauge freedom to achieve the symmetry of $T^{\mu\nu}$. We proceed from the observation that

$$T^{\mu\nu} - T^{\nu\mu} = 0 \quad \text{entails} \quad \mathcal{T}^{\mu\nu} - \mathcal{T}^{\nu\mu} + \partial_\sigma(H^{\sigma\mu\nu} - H^{\sigma\nu\mu}) = 0$$

[44] "On the spin angular momentum of mesons," Physica **7**, 882 (1939).

which by (78) becomes

$$\partial_\sigma(H^{\sigma\mu\nu} - H^{\sigma\nu\mu}) = c\,\partial_\sigma S^{\sigma\mu\nu}$$

These derivative conditions would certainly be satisfied if $H^{\sigma\mu\nu}$ satisfied the algebraic conditions

$$H^{\sigma\mu\nu} - H^{\sigma\nu\mu} = c\,S^{\sigma\mu\nu} \tag{82}$$

But $H^{\sigma\nu\mu} = -H^{\nu\sigma\mu}$ so we have the first of the following equations (the other two equations—redundant with the first—are got by cyclic permutation of indices):

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} H^{\mu\nu\sigma} \\ H^{\sigma\mu\nu} \\ H^{\nu\sigma\mu} \end{pmatrix} = c \begin{pmatrix} S^{\sigma\mu\nu} \\ S^{\nu\sigma\mu} \\ S^{\mu\nu\sigma} \end{pmatrix}$$

By matrix inversion we obtain

$$\begin{pmatrix} H^{\mu\nu\sigma} \\ H^{\sigma\mu\nu} \\ H^{\nu\sigma\mu} \end{pmatrix} = \tfrac{1}{2}c \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} S^{\sigma\mu\nu} \\ S^{\nu\sigma\mu} \\ S^{\mu\nu\sigma} \end{pmatrix}$$

Thus by a little trickery (the "permutation trick," which we will have occasion to use again in quite another connection) do we obtain

$$H^{\sigma\mu\nu} = \tfrac{1}{2}c\big\{S^{\sigma\mu\nu} + S^{\mu\nu\sigma} - S^{\nu\sigma\mu}\big\} \tag{83}$$

Since $S^{\mu\alpha\beta} \equiv \frac{1}{\mathrm{c}}\pi^\mu_a B^a{}_b{}^{\alpha\beta}\varphi^b$ (here $\pi^\mu_a \equiv \partial\mathcal{L}/\partial\varphi^a_{,\mu}$) is antisymmetric in the last pair of superscripts, the expression on the right can be written (and in the literature is written) in a variety of alternative ways; from $S^{\sigma\mu\nu} = -S^{\sigma\nu\mu}$ it follows in particular that $H^{\sigma\mu\nu} = -H^{\mu\sigma\nu}$, as was stipulated at (80.2).

Returning with (83) to (81) we have Belinfante's

$$\begin{aligned} \mathcal{T}^{\mu\nu} \longrightarrow T^{\mu\nu} &= \mathcal{T}^{\mu\nu} + \vartheta^{\mu\nu} \\ \vartheta^{\mu\nu} &\equiv \partial_\sigma H^{\sigma\mu\nu} \end{aligned} \tag{84}$$

and look now to what that adjustment does to the angular momentum tensor: we have

$$\begin{aligned} \mathcal{J}^{\mu\alpha\beta} &= \tfrac{1}{\mathrm{c}}(x^\alpha\,\mathcal{T}^{\mu\beta} - x^\beta\,\mathcal{T}^{\mu\alpha}) + S^{\mu\alpha\beta} \\ &= \tfrac{1}{\mathrm{c}}(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}) - \tfrac{1}{\mathrm{c}}\underbrace{(x^\alpha\,\vartheta^{\mu\beta} - x^\beta\,\vartheta^{\mu\alpha})} + S^{\mu\alpha\beta} \end{aligned}$$

$$\begin{aligned} &= x^\alpha\partial_\sigma H^{\sigma\mu\beta} - x^\beta\partial_\sigma H^{\sigma\mu\alpha} \\ &= \partial_\sigma(x^\alpha H^{\sigma\mu\beta} - x^\beta H^{\sigma\mu\alpha}) - \underbrace{(H^{\alpha\mu\beta} - H^{\beta\mu\alpha})} \end{aligned}$$

$$\begin{aligned} &= -(H^{\mu\alpha\beta} - H^{\mu\beta\alpha}) \\ &= -c\,S^{\mu\alpha\beta} \quad \text{by (82)} \end{aligned}$$

which (notice that the $S^{\mu\alpha\beta}$-term drops away) can be expressed

$$\mathcal{J}^{\mu\alpha\beta} \longrightarrow J^{\mu\alpha\beta} = \mathcal{J}^{\mu\alpha\beta} + \partial_\sigma(x^\alpha H^{\sigma\mu\beta} - x^\beta H^{\sigma\mu\alpha}) \tag{85}$$

The statements $\partial_\mu \mathcal{J}^{\mu\alpha\beta} = 0$ and $\partial_\mu J^{\mu\alpha\beta} = 0$ are now clearly equivalent, and the latter can be looked upon as a corollary (by $T^{\mu\nu} = T^{\nu\mu}$) of

$$J^{\mu\alpha\beta} \equiv \tfrac{1}{c}(x^\alpha\, T^{\mu\beta} - x^\beta\, T^{\mu\alpha}) \tag{86}$$

Evidently $\partial_\mu \mathcal{J}^{\mu\alpha\beta} = 0$ and $\partial_\mu J^{\mu\alpha\beta} = 0$ speak in distinct local dialects about the global conservation of the same six things

$$\iiint \mathcal{J}^{0\alpha\beta}\, dx^1 dx^2 dx^3 = \iiint J^{0\alpha\beta}\, dx^1 dx^2 dx^3 \tag{87.1}$$

as do $\partial_\mu \mathcal{T}^{\mu\alpha} = 0$ and $\partial_\mu T^{\mu\alpha\beta} = 0$ speak in distinct local dialects about the global conservation of the same four things

$$\iiint \mathcal{T}^{0\alpha}\, dx^1 dx^2 dx^3 = \iiint T^{0\alpha}\, dx^1 dx^2 dx^3 \tag{87.2}$$

Notice that the previous definition $\mathcal{M}^{\mu\alpha\beta} \equiv \frac{1}{c}(x^\alpha \mathcal{T}^{\mu\beta} - x^\beta \mathcal{T}^{\mu\alpha})$ of "orbital angular momentum" displayed momental structure, but contained no reference to the internal transformation (or spin) properties of the field system. Spin structure (see again (84)) was, however, folded into the design of $T^{\mu\nu}$, and at (86) was incorporated also into the momental design of $J^{\mu\alpha\beta}$. Introduction of (84) into (86) gives

$$\begin{aligned} J^{\mu\alpha\beta} &= M^{\mu\alpha\beta} + S^{\mu\alpha\beta} \\ M^{\mu\alpha\beta} &\equiv \mathcal{M}^{\mu\alpha\beta} + \tfrac{1}{c}\partial_\sigma(x^\alpha H^{\sigma\mu\beta} - x^\beta H^{\sigma\mu\alpha}) \end{aligned} \tag{88}$$

Belinfante's procedure[45] leaves the spin tensor $S^{\mu\alpha\beta}$ unchanged, but at the cost

$$\mathcal{M}^{\mu\alpha\beta} \longrightarrow M^{\mu\alpha\beta} = \mathcal{M}^{\mu\alpha\beta} + \tfrac{1}{c}\partial_\sigma(x^\alpha H^{\sigma\mu\beta} - x^\beta H^{\sigma\mu\alpha}) \tag{89}$$

of mixing some spin structure into the orbital angular momentum tensor; this, however, has been accomplished in such a way as to leave the (generally not conserved) global expressions

$$\iiint \mathcal{M}^{0\alpha\beta}\, dx^1 dx^2 dx^3$$

[45] The procedure is known in some circles as "symmetrization," in others as "regularization," and is frequently employed without attribution to Belinfante. The symmetrized (or regularized) stress energy tensor $T^{\mu\nu}$ is, by some fastidious authors who wish to underscore the fact that it incorporates spin structure, occasionally called the "spin-stress-energy tensor."

unaltered.

It might seem fair to ask—though the question remains unasked in the literature known to me[46]—for the $\mathcal{G}^\sigma(\varphi)$ which when introduced into (79.1) would set Belinfante's train of argument into prefigured motion. I do not know the answer, or even whether such a $\mathcal{G}^\sigma(\varphi)$ exists, but do not pursue the matter because I think deeper insight into the essence of Belinfante's procedure is, in fact, to be found elsewhere. Heretofore we have presumed that

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + \delta\omega^\mu \\ \varphi_a &\longrightarrow \varphi_a \end{aligned}$$

serves to describe the translation map in spacetime, and have by Noether's theorem been led from this presumption to the familiar stress-energy tensor[47]

$$\mathcal{T}^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial\varphi_{a,\mu}}\varphi_{a,\nu} - \mathcal{L}\delta^\mu{}_\nu$$

Suppose, however, we were—in the spirit of (1–37)—to adopt this expanded conception of "translation:"

$$\left.\begin{aligned} x^\mu &\longrightarrow x^\mu + g^{\mu\rho}\cdot\delta\omega_\rho \\ \varphi_a &\longrightarrow \varphi_a \\ \mathcal{L} &\longrightarrow \mathcal{L} - \partial_\sigma\mathcal{G}^{\sigma\rho}(\varphi)\cdot\delta\omega_\rho \end{aligned}\right\} \tag{90}$$

Noether's theorem, correspondingly expanded, would then according to (1–38) supply

$$T^{\mu\nu} = \mathcal{T}^{\mu\nu} + \mathcal{G}^{\mu\nu}$$

which would reproduce (84) if we were to set

$$\mathcal{G}^{\mu\nu} = \vartheta^{\mu\nu} \equiv \partial_\tau H^{\tau\mu\nu}$$

We are led thus from the generic (90) to what might be called "Belinfante's translation map"

$$\left.\begin{aligned} x^\mu &\longrightarrow x^\mu + g^{\mu\rho}\cdot\delta\omega_\rho \\ \varphi_a &\longrightarrow \varphi_a \\ \mathcal{L} &\longrightarrow \mathcal{L} + \partial_\sigma\partial_\tau H^{\sigma\tau\rho}\cdot\delta\omega_\rho \end{aligned}\right\} \tag{91}$$

[46] My sources have been W. Pauli, "Relativistic field theories of elementary particles," Rev. Mod. Phys. **13**, 203 (1941), §2; E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (1953), §19—Corson, by the way, combined careers in theoretical/mathematical physics and psychiatry! —and J. M. Jauch's appendix to G. Wentzel, *Quantum Theory of Fields* (1949). More recently the subject has been treated with expository freshness by D. E. Soper, *Classical Field Theory* (1976), pp. 116–123.

[47] See again (1–34).

which can be looked upon as the seed from which Belinfante's symmetrization procedure—in its entirety—spontaneously springs. Here $H^{\sigma\tau\rho}$ is understood to have the specific meaning indicated at (83), which can be written

$$H^{\sigma\tau\rho} = \tfrac{1}{2}\big\{\pi_a^\sigma B^a{}_b{}^{\tau\rho} + \pi_a^\tau B^a{}_b{}^{\rho\sigma} - \pi_a^\rho B^a{}_b{}^{\sigma\tau}\big\}\varphi^b \tag{92}$$

We note in passing that the gauge terms

$$\partial_\sigma\partial_\tau H^{\sigma\tau\rho} = 0 \quad \text{numerically}$$

and that (92) is in one (slight?) respect problematic.[48]

The intrusion of a gauge component into the definition of a map is certainly not unprecedented,[49] but the occurance of an allusion to *spin structure* in the definition of an infinitesimal *translation* is a bit of a surprise; the translational and Lorentzian parts of the Poincaré group have in (91) become fused.

Belinfante's procedure works for *any* relativistic Lagrangian field theory. It owes its success, however, not so much to relativity *per se* as to the spin structure which relativity brings to the physics. Davison Soper[46] has described a non-relativistic analog of Belinfante's procedure which again draws upon spin structure, but presumes only *rotational* invariance of the Lagrangian.

What—if any—*physical* significance are we to associate with the distinction between

$$\mathfrak{T}^{\mu\alpha} \quad \text{and} \quad T^{\mu\alpha} = \mathfrak{T}^{\mu\alpha} + (\text{divergenceless gauge term})^{\mu\alpha} \tag{93}$$

It is sometimes argued in electrodynamics that the field $F^{\mu\nu}$ announces itself never directly, but always indirectly—through the associated stress-energy tensor, which mediates the interaction of field with charged matter. But when one looks to the equation which describes that interaction[50] one observes that it is of a form

$$\partial_\mu(\text{stress-energy of field})^{\mu\nu} + \partial_\mu(\text{stress-energy of matter})^{\mu\nu} = 0$$

---

[48] Looking into the abbreviation $\pi_a^\mu \equiv \partial\mathcal{L}/\partial\varphi^a{}_{,\mu} = \pi_a^\mu(\varphi, \partial\varphi)$, we note that $\partial\varphi$-dependence is unwelcome in a gauge term. The problem does not arise, however, in *canonically* formulated theories (for the same reason that they do not admit of Hamiltonian formulation), and I must on this occasion be content to let it drop. But look to the footnote on p. 121 of Soper.[46]

[49] Construction of a Lagrangian account of the Galilean-covariant systems of particles is found to entail incorporation of a gauge transformation into the definition of a Galilean transformation, and in the case $L = \frac{1}{2}m\dot{x}^2$ leads to a conservation law formally identical to the previously encountered conservation of $\boldsymbol{K} = t\boldsymbol{P} - M\boldsymbol{X}$; see CLASSICAL MECHANICS (1983), p. 169. A similar detail intrudes into the quantum theory of such systems.

[50] See CLASSICAL ELECTRODYNAMICS (1980), p. 312.

into which the stress-energy tensor enters not nakedly, but differentiated. In such contexts the distinction (93) is without consequence. We conclude that transformations of the form

$$\mathfrak{J}_r^{\mu} \longrightarrow J_r^{\mu} = \mathfrak{J}_r^{\mu} + (\text{divergenceless gauge term})_r^{\mu}$$

can be considered "physical"—and thus to present a selection problem—only in contexts (should any present themselves) into which the Noetherean current in question enters "nakedly." General relativity presents a context into which the stress-energy tensor does just that, for it is *via* its stress-energy tensor—the repository of information concerning the *distribution* of field-energy/mass—that a field system generates/feels gravitational fields. In that context "the dialect makes a difference." Several lines of argument indicate that it is the symmetrized stress-energy tensor $T^{\mu\nu}$—not $\mathfrak{T}^{\mu\nu}$—which speaks the preferred dialect when stripped naked by gravity.[51]

Belinfante's brilliant little mathematical dance is, in my opinion, classic; it uses the simplest and most general of means to achieve a far-reaching—and conceptually challenging—result of great practical importance. Because he had a local history, I allow myself this sentimental digression: Frederic Jozef Belinfante was born in 1913 in The Hague. He took his doctorate in 1939 from the University of Leiden, where his research was directed by H. A. Kramers, and it is (I infer) in his dissertation that the symmetrization procedure—the work of a 26-year-old—was first described; the thesis itself presented "the germ of the connection between the 'spin-statistics theorem' and the yet undiscovered TCP theorem,"[52] and is for that reason still highly regarded. Belinfante was professionally inactive during the war years (which he spent in Holland), but in 1946 accepted a faculty appointment to the University of British Columbia, and in 1948 went to Purdue, from which he retired in 1979. Belinfante had wide-ranging interests outside of physics; he was, like many of his countrymen, a linguist (wrote scientific papers in Esperanto, and introduced—in addition to a great many failed neologisms—the word "nucleon" into the vocabulary of physics), and found endless fascination in maps, railway schedules, meteorological charts. It was study of the latter which led him in his retirement to Portland, which by his calculation presented the best American approximation to the climate of Holland. When in the early 1980's Jean Delord brought to my notice a letter to the editors of *Physics Today* signed by one "F. J. Belinfante, Gresham, Oregon" I immediately looked the man up in the phone book, and gave him a call. Soon I was able to arrange for him to be named Adjunct Professor (Reed's first) and provided with an office, a computer (his

[51] See Léon Rosenfeld, "Sur le tenseur d'impulse-energie," Memoires de l'Academie Roy. Belgique **6**, 30 (1940). Also C. W. Misner, K. S. Thorne & J. A. Wheeler, *Gravitation* (1973), §21.3.

[52] I quote from I. Duck & E. C. G. Sudarshan, *Pauli and the Spin-Statistics Theorem* (1997), p. 301. Chapter 13 of that monograph treats "Belinfante's Proof of the Spin-Statistics Theorem." Belinfante's work led first to a dispute—but later to a collaborative paper (1940)—with Pauli.

first, though when he arrived in Portland he had wasted no time in joining the local HP–45 Club), and people to talk to. Throughout the 1980's he was seen—a slight, grandfatherly man in a large coat, trailing his briefcase on a little cart (a tradition now carried forward by me)—as he made his way from the East Parking Lot to the Physics Building. He gave us some seminars, consumed many of David Griffiths' afternoons discussing computational problems at the blackboard (he loved to talk, had lots of complicated stuff on his mind, and was oblivious of the possibility that David might have other things to do), and busied himself with the simultaneous preparation of two projected books in the tradition of his *A Survey of Hidden-Variable Theories* (1973); one of those was to treat quantum field theory, the other general relativity, and both remained unfinished at the time of his death, in the early summer of 1991. He left his personal library to Reed College. We talked of many things on many occasions. I regret that I never talked with him about the circumstances which led to the invention of his symmetrization procedure.[53]

Conservation laws for some illustrative field systems. My objective here will be to show how the general results developed in the preceding two sections look when brought to ground in particular cases. The conservation laws we will be looking at all derive from the postulated *Poincaré covariance* of the field systems in question.

REAL SCALAR FIELD

Here some of the main results lie already at hand;[54] I write them out again to place them in their larger context, to establish some notation, and to set the pattern to which I will adhere when discussing more complicated systems. The Lagrangian reads

$$\mathcal{L} = \tfrac{1}{2}\big\{g^{\rho\sigma}\varphi_{,\rho}\varphi_{,\sigma} - \varkappa^2\varphi^2\big\} \tag{11 \equiv 94.0}$$

The stress-energy tensor derives from the translational component of the Poincaré group; infinitesimally

$$\left.\begin{aligned} x^\mu &\longrightarrow x^\mu + \mathcal{X}^{\mu\alpha}\cdot\delta\omega_\alpha \quad\text{with}\quad \mathcal{X}^{\mu\alpha} = g^{\mu\alpha}\\ \varphi &\longrightarrow \varphi \;+\; \varPhi^\alpha\cdot\delta\omega_\alpha \quad\text{with}\quad \varPhi^\alpha = 0 \end{aligned}\right\} \tag{94.1}$$

which when fed into the generic

$$\begin{aligned} J^{\mu\alpha} &= \pi^\mu\big\{\varPhi^\alpha - \varphi_{,\sigma}\mathcal{X}^{\sigma\alpha}\big\} + \mathcal{L}\mathcal{X}^{\mu\alpha}\\ \pi^\mu &\equiv \partial\mathcal{L}/\partial\varphi_{,\mu} \end{aligned} \tag{94.2}$$

[53] For further information see the obituary printed on p. 82 of the July 1992 issue of *Physics Today*. Related material can also be found in M. Dresden, *H. A. Kramers: Between Tradition and Revolution* (1987).

[54] See again the material subsequent to (11). Also (1–41). The funny equation numbers are synchronized to those encountered in subsequent examples.

gives (after the conventional sign reversal)

$$\begin{aligned}\mathcal{T}^{\mu\alpha} &= -\Big[g^{\mu\rho}\varphi_{,\rho}\big\{0-\varphi_{,\sigma}g^{\sigma\alpha}\big\}+\mathcal{L}g^{\mu\alpha}\Big]\\ &= \pi^\mu\pi^\alpha-\tfrac{1}{2}\big(g_{\rho\sigma}\pi^\rho\pi^\sigma-\varkappa^2\varphi^2\big)g^{\mu\alpha}\end{aligned}\tag{94.10}$$

where $\pi^\mu = \partial^\mu\varphi$ in the case at hand. The stress-energy tensor is already symmetric

$$\mathcal{T}^{\mu\alpha}=\mathcal{T}^{\alpha\mu}\tag{94.11}$$

and the energy density is given by

$$\mathcal{E}=\mathcal{T}^{00}=\tfrac{1}{2}\big\{\pi^0\pi^0+\pi^1\pi^1+\pi^2\pi^2+\pi^3\pi^3+\varkappa^2\varphi^2\big\}\geqslant 0\tag{94.12}$$

The (proper) Lorentz component of the Poincaré group gives[55]

$$\left.\begin{aligned} x^\mu &\longrightarrow x^\mu+\tfrac{1}{2}\mathcal{X}^{\mu\alpha\beta}\cdot\delta\Omega_{\alpha\beta}\quad\text{with}\quad \mathcal{X}^{\mu\alpha\beta}=(g^{\mu\alpha}x^\beta-g^{\mu\beta}x^\alpha)\\ \varphi &\longrightarrow \varphi\ +\tfrac{1}{2}\ \Phi^{\alpha\beta}\cdot\delta\Omega_{\alpha\beta}\quad\text{with}\quad \Phi^{\alpha\beta}=0\end{aligned}\right\}\tag{94.4}$$

which when fed into the generic

$$J^{\mu\alpha\beta}=\pi^\mu\big\{\Phi^{\alpha\beta}-\varphi_{,\sigma}\mathcal{X}^{\sigma\alpha\beta}\big\}+\mathcal{L}\mathcal{X}^{\mu\alpha\beta}\tag{94.5}$$

give (compare (76))

$$\begin{aligned}\mathcal{J}^{\mu\alpha\beta}&=\tfrac{1}{c}\Big[\big(-\pi^\mu\pi^\alpha+\mathcal{L}g^{\mu\alpha}\big)x^\beta-\big(-\pi^\mu\pi^\beta+\mathcal{L}g^{\mu\beta}\big)x^\alpha\Big]\\ &=\tfrac{1}{c}\big(x^\alpha\mathcal{T}^{\mu\beta}-x^\beta\mathcal{T}^{\mu\alpha}\big)+\text{no spin component}\end{aligned}\tag{94.16}$$

In the absence of a spin component Belinfante's procedure has nothing to work with, but on the other hand it has nothing to do: stress-energy symmetry ensures

$$\partial_\mu\mathcal{J}^{\mu\alpha\beta}=0$$

Similar results are obtain in the case

$$\mathcal{L}=\tfrac{1}{2}\big\{g^{\rho\sigma}\psi^*_{,\rho}\psi_{,\sigma}-\varkappa^2\psi^*\psi\big\}\tag{19}$$

of a complex scalar field.

PROCA THEORY

I refer under that heading to the real vector field system

$$\begin{aligned}\mathcal{L}&=\underbrace{\tfrac{1}{4}(U^{\rho,\sigma}-U^{\sigma,\rho})(U_{\rho,\sigma}-U_{\sigma,\rho})}-\tfrac{1}{2}\varkappa^2U^\sigma U_\sigma \qquad (28.2\equiv 95.0)\\ &\qquad\qquad\qquad\qquad\quad =\tfrac{1}{2}(U^{\rho,\sigma}U_{\rho,\sigma}-U^{\rho,\sigma}U_{\sigma,\rho})\end{aligned}$$

[55] See again (75).

into the theory of which the antisymmetric tensor field $G^{\mu\nu} \equiv \partial^\mu U^\nu - \partial^\nu U^\mu$ enters simply as a notational device. Introducing

$$\left.\begin{aligned} x^\mu &\longrightarrow x^\mu + \mathcal{X}^{\mu\alpha}\cdot\delta\omega_\alpha \quad \text{with} \quad \mathcal{X}^{\mu\alpha} = g^{\mu\alpha} \\ U^\mu &\longrightarrow U^\mu + \Phi^{\mu\alpha}\cdot\delta\omega_\alpha \quad \text{with} \quad \Phi^{\mu\alpha} = 0 \end{aligned}\right\} \tag{95.1}$$

into the generic

$$\begin{aligned} J^{\mu\alpha} =& \pi^{\mu\nu}\big\{\Phi_\nu{}^\alpha - U_{\nu,\sigma}\mathcal{X}^{\sigma\alpha}\big\} + \mathcal{L}\mathcal{X}^{\mu\alpha} \\ & \pi^{\mu\nu} \equiv \partial\mathcal{L}/\partial U_{\nu,\mu} \end{aligned} \tag{95.2}$$

we notice that $\pi^{\mu\nu} = U^{\nu,\mu} - U^{\mu,\nu} \equiv G^{\mu\nu}$ in the present instance, and that the Lagrangian can be notated $\mathcal{L} = \frac{1}{4}\pi^{\rho\sigma}\pi_{\rho\sigma} - \frac{1}{2}\varkappa^2 U^\sigma U_\sigma$, and so obtain

$$\begin{aligned} \mathcal{T}^{\mu\alpha} &= \pi^\mu{}_\nu U^{\nu,\alpha} - \mathcal{L}g^{\mu\alpha} \\ &\neq \mathcal{T}^{\alpha\mu} \end{aligned} \tag{95.3}$$

The infinitesimal Lorentz map reads

$$\left.\begin{aligned} x^\mu &\longrightarrow x^\mu + \tfrac{1}{2}\mathcal{X}^{\mu\alpha\beta}\cdot\delta\Omega_{\alpha\beta} \quad \text{with} \quad \mathcal{X}^{\mu\alpha\beta} = (g^{\mu\alpha}\,x^\beta - g^{\mu\beta}\,x^\alpha) \\ U^\mu &\longrightarrow U^\mu + \tfrac{1}{2}\Phi^{\mu\alpha\beta}\cdot\delta\Omega_{\alpha\beta} \quad \text{with} \quad \Phi^{\mu\alpha\beta} = (g^{\mu\alpha}U^\beta - g^{\mu\beta}U^\alpha) \end{aligned}\right\} \tag{95.4}$$

which when introduced into the generic

$$J^{\mu\alpha\beta} = \pi^{\mu\nu}\big\{\Phi_\nu{}^{\alpha\beta} - U_{\nu,\sigma}\mathcal{X}^{\sigma\alpha\beta}\big\} + \mathcal{L}\mathcal{X}^{\mu\alpha\beta} \tag{95.5}$$

give

$$\begin{aligned} \mathcal{J}^{\mu\alpha\beta} =& \ \tfrac{1}{c}\Big[\big(-\pi^\mu{}_\nu U^{\nu,\alpha} + \mathcal{L}g^{\mu\alpha}\big)x^\beta - \big(-\pi^\mu{}_\nu U^{\nu,\beta} + \mathcal{L}g^{\mu\beta}\big)x^\alpha\Big] \\ & + \tfrac{1}{c}\Big[\pi^{\mu\alpha}U^\beta - \pi^{\mu\beta}U^\alpha\Big] \\ =& \tfrac{1}{c}\big(x^\alpha\mathcal{T}^{\mu\beta} - x^\beta\mathcal{T}^{\mu\alpha}\big) + S^{\mu\alpha\beta} \end{aligned} \tag{95.6}$$

$$S^{\mu\alpha\beta} = \tfrac{1}{c}\big(G^{\mu\alpha}U^\beta - G^{\mu\beta}U^\alpha\big) \tag{95.7}$$

We have now in hand all we need to accomplish the *symmetrization* of $\mathcal{T}^{\mu\alpha}$. The generic equation (83) becomes in the present instance

$$\begin{aligned} H^{\sigma\mu\nu} &= \tfrac{1}{2}c\big\{\mathcal{S}^{\sigma\mu\nu} + \mathcal{S}^{\mu\nu\sigma} - \mathcal{S}^{\nu\sigma\mu}\big\} \\ &= \tfrac{1}{2}\big\{G^{\sigma\mu}U^\nu - G^{\sigma\nu}U^\mu + G^{\mu\nu}U^\sigma - G^{\mu\sigma}U^\nu - G^{\nu\sigma}U^\mu + G^{\nu\mu}U^\sigma\big\} \\ &= G^{\sigma\mu}U^\nu \quad \text{after cancellations resulting from } G^{\mu\nu} = -G^{\nu\mu} \end{aligned} \tag{95.8}$$

Belinfante's (84) now gives

$$\begin{aligned} \mathcal{T}^{\mu\alpha} \longrightarrow T^{\mu\alpha} = \mathcal{T}^{\mu\alpha} &+ \partial_\sigma H^{\sigma\mu\alpha} \\ \partial_\sigma H^{\sigma\mu\alpha} &= U^\alpha\partial_\sigma G^{\sigma\mu} + G^{\sigma\mu}\partial_\sigma U^\alpha \end{aligned} \tag{95.9}$$

Drawing upon the field equations and (once again) upon $G^{\sigma\mu} = -G^{\mu\sigma}$, we have

$$= -\varkappa^2 U^\alpha U^\mu - G^\mu{}_\sigma U^{\alpha,\sigma}$$

giving

$$= G^\mu{}_\sigma U^{\sigma,\alpha} - \mathcal{L} g^{\mu\alpha} - \varkappa^2 U^\mu U^\alpha - G^\mu{}_\sigma U^{\alpha,\sigma}$$

whence

$$T^{\mu\alpha} = G^\mu{}_\sigma G^{\alpha\sigma} - \mathcal{L} g^{\mu\alpha} - \varkappa^2 U^\mu U^\alpha \tag{95.10}$$

which is manifestly symmetric:

$$T^{\mu\alpha} = T^{\alpha\mu} \tag{95.11}$$

The implied energy density function is

$$\begin{aligned}
\mathcal{E} = T^{00} &= G^0{}_\sigma G^{0\sigma} - \big(\tfrac{1}{4}G^{\sigma\rho}G_{\sigma\rho} - \tfrac{1}{2}\varkappa^2 U^\sigma U_\sigma\big) - \varkappa^2 U^0 U^0 \\
&= \big(-G_{01}^2 - G_{02}^2 - G_{03}^2\big) \\
&\qquad + \tfrac{1}{2}\big(G_{01}^2 + G_{02}^2 + G_{03}^2 - G_{12}^2 - G_{23}^2 - G_{31}^2\big) \\
&\qquad + \tfrac{1}{2}\varkappa^2\big(U_0^2 - U_1^2 - U_2^2 - U_3^2\big) - \varkappa^2 U_0^2 \\
&= -\tfrac{1}{2}\Big[\big(G_{01}^2 + G_{02}^2 + G_{03}^2 + G_{12}^2 + G_{23}^2 + G_{31}^2\big) \\
&\qquad\qquad + \varkappa^2\big(U_0^2 + U_1^2 + U_2^2 + U_3^2\big)\Big] \leqslant 0
\end{aligned} \tag{95.12}$$

Evidently the

Procca theory requires a final sign-reversal to bring the regularization process to completion:

$$\begin{aligned}
T^{\mu\alpha} \longrightarrow \mathring{T}^{\mu\alpha} &\equiv -T^{\mu\alpha} \\
&= G^\mu{}_\sigma G^{\sigma\alpha} + \mathcal{L} g^{\mu\alpha} + \varkappa^2 U^\mu U^\alpha
\end{aligned} \tag{95.13}$$

This step could be averted if we backed up to (95.0) and flipped the sign of the Lagrangian, but such a step might seem unmotivated; the moral is: *The sign of the Lagrangian makes a difference—not to the equations of motion, but to some of their deeper formal ramifications*. Note also that while $\mathcal{T}^{00}$ and T^{00} assign identical values to the *total* energy, they differ by a term

$$\begin{aligned}
\partial_\sigma H^{\sigma 00} &= -\varkappa^2 U^0 U^0 - G^0{}_\sigma U^{0,\sigma} \\
&= -\varkappa^2 U_0^2 + \big(G_{01}U_{0,1} + G_{02}U_{0,2} + G_{03}U_{0,3}\big) \\
&= -\varkappa^2 U_0^2 - \big(U_{0,1}^2 + U_{0,2}^2 + U_{0,3}^2\big) + \big(U_{1,0}U_{0,1} + U_{2,0}U_{0,2} + U_{3,0}U_{0,3}\big) \\
&= -\varkappa^2 U_0^2 - \tfrac{1}{2}\big(U_{0,1}^2 + U_{0,2}^2 + U_{0,3}^2\big) - \tfrac{1}{2}\big(G_{01}^2 + G_{02}^2 + G_{03}^2\big) \\
&\qquad + \tfrac{1}{2}\big(U_{1,0}^2 + U_{2,0}^2 + U_{3,0}^2\big)
\end{aligned}$$

which is not only non-zero but *sign-indefinite*:[56] prior to regularization we stood at risk of violating the principle that energy density must be non-negative. Spin

[56] In the preceding argument I have used

$$G_{\mu\nu}^2 = (U_{\nu,\mu} - U_{\mu,\nu})^2 = U_{\mu,\nu}^2 + U_{\nu,\mu}^2 - 2U_{\mu,\nu}U_{\nu,\mu} \geqslant 0$$

structure was incorporated into the design of $\mathring{T}^{\mu\alpha}$, and is therefore incorporated also into the momental design of

$$\mathring{J}^{\mu\alpha\beta} \equiv \tfrac{1}{c}(x^\alpha \mathring{T}^{\mu\beta} - x^\beta \mathring{T}^{\mu\alpha}) \tag{95.14}$$

Working from the generic equation (82) we are led back again to the spin tensor

$$\begin{aligned}\mathring{S}^{\mu\alpha\beta} = -S^{\mu\alpha\beta} &= \tfrac{1}{c}\big(H^{\alpha\mu\beta} - H^{\beta\mu\alpha}\big)\\ &= \tfrac{1}{c}\big(G^{\alpha\mu}U^\beta - G^{\beta\mu}U^\alpha\big)\end{aligned} \tag{95.15}$$

previously encountered at (95.7). The definition of the orbital angular momentum tensor is implicit in the relation

$$\mathring{J}^{\mu\alpha\beta} = \mathring{M}^{\mu\alpha\beta} + \mathring{S}^{\mu\alpha\beta} \tag{95.16}$$

$\partial_\mu \mathring{M}^{\mu\alpha\beta}$ and $\partial_\mu \mathring{S}^{\mu\alpha\beta}$ both fail to vanish, but in such a concerted way as to achieve $\partial_\mu \mathring{J}^{\mu\alpha\beta} = 0$.

ELECTRODYNAMICS

We look upon Maxwellian electrodynamics as "Procca theory in the zero-mass limit $\varkappa \downarrow 0$." By way of notational preparation, let us agree to write[57]

$$\begin{pmatrix} U^0\\ U^1\\ U^2\\ U^3\end{pmatrix} = \begin{pmatrix}\phi\\ \boldsymbol{\mathfrak{A}}\end{pmatrix} \quad\text{and}\quad \|G^{\mu\nu}\| = \begin{pmatrix} 0 & -\mathfrak{E}_1 & -\mathfrak{E}_2 & -\mathfrak{E}_3\\ \mathfrak{E}_1 & 0 & -\mathfrak{B}_3 & \mathfrak{B}_2\\ \mathfrak{E}_2 & \mathfrak{B}_3 & 0 & -\mathfrak{B}_1\\ \mathfrak{E}_3 & -\mathfrak{B}_2 & \mathfrak{B}_1 & 0\end{pmatrix} \tag{96}$$

The Procca field equations (32)

$$\begin{gathered} G^{\mu\nu} = \partial^\mu U^\nu - \partial^\nu U^\mu\\ \partial^\lambda G^{\mu\nu} + \partial^\mu G^{\nu\lambda} + \partial^\nu G^{\lambda\mu} = 0\\ \partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu = 0\\ \partial_\nu U^\nu = 0\end{gathered}$$

can in this notation be expressed

$$\begin{gathered} \boldsymbol{\mathfrak{E}} = -\boldsymbol{\nabla}\phi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{A}} \qquad\text{and}\qquad \boldsymbol{\mathfrak{B}} = \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{A}}\\ \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{E}} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{B}} = \mathbf{0} \qquad\text{and}\qquad \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{B}} = 0\\ \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{E}} = -\varkappa^2\phi \qquad\text{and}\qquad \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{B}} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{E}} = -\varkappa^2\boldsymbol{\mathfrak{A}}\\ \tfrac{1}{c}\tfrac{\partial}{\partial t}\phi + \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{A}} = 0\end{gathered}$$

In the limit $\varkappa \downarrow 0$ these equations assume precisely the form of the *Maxwell's*

(continued from the preceding page) to obtain

$$\begin{aligned}\big(U_{1,0}U_{0,1} + U_{2,0}U_{0,2} + U_{3,0}U_{0,3}\big) &= \tfrac{1}{2}\big(U_{0,1}^2 + U_{0,2}^2 + U_{0,3}^2 + U_{1,0}^2 + U_{2,0}^2 + U_{3,0}^2\big)\\ &\qquad - \tfrac{1}{2}\big(G_{01}^2 + G_{02}^2 + G_{03}^2\big)\\ &\leqslant \tfrac{1}{2}\big(U_{0,1}^2 + U_{0,2}^2 + U_{0,3}^2 + U_{1,0}^2 + U_{2,0}^2 + U_{3,0}^2\big)\end{aligned}$$

[57] Compare CLASSICAL ELECTRODYNAMICS (1980), pp. 162 & 373.

equations in the absence of sources,[58] joined by the Lorentz gauge condition as an automatic tag-along. We have

$$\begin{aligned}\mathring{T}^{00} &= \text{energy density}\\ &= \tfrac{1}{2}\big\{\mathfrak{E}^2 + \mathfrak{B}^2 + \varkappa^2\big(\phi^2 + \mathfrak{A}^2\big)\big\} \qquad (97.1)\\ \tfrac{1}{c}\begin{pmatrix}\mathring{T}^{01}\\ \mathring{T}^{02}\\ \mathring{T}^{03}\end{pmatrix} &= \text{momentum density vector } \boldsymbol{P}\\ &= \tfrac{1}{c}\big\{\boldsymbol{\mathfrak{E}}\times\boldsymbol{\mathfrak{B}} + \varkappa^2\phi\mathfrak{A}\big\} \qquad (97.2)\end{aligned}$$

which again give back familiar results in the limit $\varkappa \downarrow 0$. Less familiar is the construction (see again (95.15))

$$\begin{aligned}\tfrac{1}{c}\begin{pmatrix}\mathring{S}^{023}\\ \mathring{S}^{031}\\ \mathring{S}^{012}\end{pmatrix} = \tfrac{1}{c}\begin{pmatrix}G^{20}U^3 - G^{30}U^2\\ G^{30}U^1 - G^{10}U^3\\ G^{10}U^2 - G^{20}U^1\end{pmatrix} &= \text{spin density vector } \boldsymbol{S}\\ &= \tfrac{1}{c}\boldsymbol{\mathfrak{E}}\times\boldsymbol{\mathfrak{A}} \qquad (98)\end{aligned}$$

This result is at least dimensionally correct,[59] but is nonetheless puzzling in this profound respect: (98) contains no allusion to the mass parameter $\varkappa$, and therefore remains unchanged in the electromagnetic limit, where we expect to obtain

$$\boldsymbol{S} = \tfrac{1}{c}\boldsymbol{E}\times\boldsymbol{A} \qquad \textit{but only in the Lorentz gauge}$$

One would like

- to develop a *manifestly gauge-invariant* description of $\boldsymbol{S}$, else
- to identify *properties* of $\boldsymbol{S}$ which are gauge-invariant (therefore physical).

But those are assignments which I must save for another occasion.[60] It is a

---

[58] Curiously, if we strike the first line and make the notational/conceptual adjustments

$$-\varkappa^2\phi \longmapsto \rho \qquad \text{and} \qquad -\varkappa^2\boldsymbol{\mathfrak{A}} \longmapsto \tfrac{1}{c}\mathbf{j}$$

we obtain

$$\begin{gathered}\boldsymbol{\nabla}\times\boldsymbol{\mathfrak{E}} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{B}} = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{B}} = 0\\ \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{E}} = \rho \qquad \text{and} \qquad \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{B}} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{E}} = \tfrac{1}{c}\mathbf{j}\\ \tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\mathbf{j} = 0\end{gathered}$$

which are precisely Maxwell's equations *in the presence* of sources. What was formerly the Lorentz gauge condition has become the charge continuity equation. The introduction of potentials ϕ and $\mathfrak{A}$ acquires now the status of an unexploited option.

[59] Working from $[G^2] = [\varkappa^2 U^2] =$ (energy density) and $[\varkappa] = (\text{length})^{-1}$ we find $[\frac{1}{c}GU] = [\frac{1}{c}\mathfrak{E}\,\mathfrak{A}] = (\text{energy}\cdot\text{time})/(\text{length})^3$ =action density.

[60] In the meantime, see Soper,[46] p. 115; Corson,[46] p. 81; or Bjørn Felsager, *Geometry, Particles & Fields* (1997), §3.6.

lesson of experience that when $\boldsymbol{A}$-potentials step nakedly onto the stage we can expect odd goings-on, and that our careful attention will be rewarded.[61]

DIRAC FIELD

We revisit the system

$$\mathcal{L} = -\hbar c\Big\{i\tfrac{1}{2}\big\{\tilde{\psi}_{,\mu}\boldsymbol{\gamma}^\mu\psi - \tilde{\psi}\boldsymbol{\gamma}^\mu\psi_{,\mu}\big\} + \varkappa\,\tilde{\psi}\psi\Big\} \tag{56 ≡ 99.0}$$

where ψ is now a 4-component spinor field. Introducing

$$\left.\begin{aligned} x^\mu &\longrightarrow x^\mu + \mathfrak{X}^{\mu\alpha}\cdot\delta\omega_\alpha \quad\text{with}\quad \mathfrak{X}^{\mu\alpha} = g^{\mu\alpha}\\ \psi^a &\longrightarrow \psi^a + \Phi^{a\alpha}\cdot\delta\omega_\alpha \quad\text{with}\quad \Phi^{a\alpha} = 0\\ \tilde{\psi}^a &\longrightarrow \tilde{\psi}^a + \tilde{\Phi}^{a\alpha}\cdot\delta\omega_\alpha \quad\text{with}\quad \tilde{\Phi}^{a\alpha} = 0 \end{aligned}\right\} \tag{99.1}$$

into the generic

$$J^{\mu\alpha} = -\Big[\pi^\mu{}_a\big\{\Phi^{a\alpha} - \psi^a{}_{,\sigma}\mathfrak{X}^{\sigma\alpha}\big\} + \big\{\tilde{\Phi}^{a\alpha} - \tilde{\psi}^a{}_{,\sigma}\mathfrak{X}^{\sigma\alpha}\big\}\tilde{\pi}^\mu{}_a + \mathcal{L}\mathfrak{X}^{\mu\alpha}\Big] \tag{99.2}$$

$$\pi^\mu{}_a \equiv \partial\mathcal{L}/\partial\psi^a{}_{,\mu} \qquad\qquad \tilde{\pi}^\mu{}_a \equiv \partial\mathcal{L}/\partial\tilde{\psi}^a{}_{,\mu}$$

we observe that

$$\pi^\mu = +\tfrac{1}{2}i\hbar c\tilde{\psi}\boldsymbol{\gamma}^\mu \qquad\text{and}\qquad \tilde{\pi}^\mu = -\tfrac{1}{2}i\hbar c\boldsymbol{\gamma}^\mu\psi$$

in the present instance, and that the Lagrangian can for the purposes at hand be notated

$$\begin{aligned}\mathcal{L} &= \pi^\sigma\psi_{,\sigma} + \tilde{\psi}_{,\sigma}\tilde{\pi}^\sigma - mc^2\tilde{\psi}\psi\\ &= 0 \quad\text{numerically, as established at (58)}\end{aligned}$$

So (dropping the $\mathcal{L}g^{\mu\alpha}$-term) we have

$$\begin{aligned}\mathfrak{T}_{\mu\alpha} &= \pi_\mu\psi_{,\alpha} + \tilde{\psi}_{,\alpha}\tilde{\pi}_\mu\\ &= \tfrac{1}{2}i\hbar c\big\{\tilde{\psi}\boldsymbol{\gamma}_\mu\psi_{,\alpha} - \tilde{\psi}_{,\alpha}\boldsymbol{\gamma}_\mu\psi\big\}\\ &\neq \mathfrak{T}_{\alpha\mu}\end{aligned} \tag{99.3}$$

We are forced therefore to look to the spin structure of the Dirac field, as a first step toward Belinfante symmetrization (which in this special case Pauli[62] had accomplished by *ad hoc* methods already in 1933). The infinitesimal Lorentz map reads

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + \tfrac{1}{2}\mathfrak{X}^{\mu\alpha\beta}\cdot\delta\Omega_{\alpha\beta} \quad\text{with}\quad \mathfrak{X}^{\mu\alpha\beta} = (g^{\mu\alpha}\,x^\beta - g^{\mu\beta}\,x^\alpha)\\ \psi^a &\longrightarrow \psi^a + \tfrac{1}{2}\Phi^{a\alpha\beta}\cdot\delta\Omega_{\alpha\beta} \quad\text{with}\quad \Phi^{a\alpha\beta} = B^a{}_b{}^{\alpha\beta}\psi^b\end{aligned}$$

---

[61] See, for example, the discussion of the Aharonov-Bohm effect in Griffiths' *Introduction to Quantum Mechanics* (1994), §10.2.4.

[62] See p. 235 in "Die allgemeinen Prinzipien der Wellenmechanik," *Handbuch der Physik* (2nd edition) **24/1** (1933), which was reprinted as a separate volume in 1950.

where preliminary information relating to the structure functions $\Phi^{a\alpha\beta}$ has been harvested from (75.2), but it remains to figure out the designs of the 4×4 matrices $\boldsymbol{B}^{\alpha\beta} \equiv \|B^a{}_b{}^{\alpha\beta}\|$; that, however, is easily accomplished, since the main work has already been done: returning with (74) to (72), we have

$$\boldsymbol{B} = \tfrac{1}{8}A^{\mu\nu}(\boldsymbol{\gamma}_\mu\boldsymbol{\gamma}_\nu - \boldsymbol{\gamma}_\nu\boldsymbol{\gamma}_\mu)$$
$$A^{\mu\nu} = \tfrac{1}{2}(g^{\mu\alpha}g^{\beta\nu} - g^{\mu\beta}g^{\alpha\nu})\delta\Omega_{\alpha\beta}$$

giving

$$\boldsymbol{B} = \tfrac{1}{2}\boldsymbol{B}^{\alpha\beta}\delta\Omega_{\alpha\beta}$$
$$\boldsymbol{B}^{\alpha\beta} = \tfrac{1}{4}(\boldsymbol{\gamma}^\alpha\boldsymbol{\gamma}^\beta - \boldsymbol{\gamma}^\beta\boldsymbol{\gamma}^\alpha)$$

From $\tilde{\psi} \equiv \psi^\dagger\boldsymbol{G}$ it follows that if $\delta_\Omega\psi = \boldsymbol{B}\psi$ then $\delta_\Omega\tilde{\psi} = \tilde{\psi}\tilde{\boldsymbol{B}}$, with $\tilde{\boldsymbol{B}} \equiv \boldsymbol{G}^{-1}\boldsymbol{B}^\dagger\boldsymbol{G}$. But the defining properties[63] of $\boldsymbol{G}$ entail $\tilde{\boldsymbol{\gamma}}^\mu = \boldsymbol{\gamma}^\mu$, from which it follows readily that $\tilde{\boldsymbol{B}}^{\alpha\beta} = \boldsymbol{B}^{\beta\alpha} = -\boldsymbol{B}^{\alpha\beta}$. The action of the infinitesimal Lorentz map can therefore be described

$$\left.\begin{array}{l} x^\mu \longrightarrow x^\mu + \tfrac{1}{2}\mathfrak{X}^{\mu\alpha\beta}\cdot\delta\Omega_{\alpha\beta} \quad\text{with}\quad \mathfrak{X}^{\mu\alpha\beta} = (g^{\mu\alpha}\,x^\beta - g^{\mu\beta}\,x^\alpha) \\ \psi \longrightarrow \psi + \tfrac{1}{2}\Phi^{\alpha\beta}\cdot\delta\Omega_{\alpha\beta} \quad\text{with}\quad \Phi^{\alpha\beta} = +\tfrac{1}{4}(\boldsymbol{\gamma}^\alpha\boldsymbol{\gamma}^\beta - \boldsymbol{\gamma}^\beta\boldsymbol{\gamma}^\alpha)\psi \\ \tilde{\psi} \longrightarrow \tilde{\psi} + \tfrac{1}{2}\tilde{\Phi}^{\alpha\beta}\cdot\delta\Omega_{\alpha\beta} \quad\text{with}\quad \tilde{\Phi}^{\alpha\beta} = -\tfrac{1}{4}\tilde{\psi}(\boldsymbol{\gamma}^\alpha\boldsymbol{\gamma}^\beta - \boldsymbol{\gamma}^\beta\boldsymbol{\gamma}^\alpha) \end{array}\right\} \tag{99.4}$$

We have achieved notational simplicity by surpressing the superscripts ^{a} that distinguish the components of the spinor field ψ, and would achieve more by introducing the $\alpha\beta$-antisymmeteric array of 4×4 matrices

$$\boldsymbol{\sigma}^{\alpha\beta} \equiv \tfrac{1}{2i}(\boldsymbol{\gamma}^\alpha\boldsymbol{\gamma}^\beta - \boldsymbol{\gamma}^\beta\boldsymbol{\gamma}^\alpha)$$

where the i has been introduced so as to achieve $\tilde{\boldsymbol{\sigma}}^{\alpha\beta} = \boldsymbol{\sigma}^{\alpha\beta}$. Returning with this information to the generic

$$J^{\mu\alpha\beta} = -\big\{\pi^\mu\psi_{,\sigma} + \tilde{\psi}_{,\sigma}\tilde{\pi}^\mu\big\}\mathfrak{X}^{\sigma\alpha\beta} + \big\{\pi^\mu\Phi^{\alpha\beta} + \tilde{\Phi}^{\alpha\beta}\tilde{\pi}^\mu\big\} \tag{99.5}$$

we obtain

$$\mathcal{J}^{\mu\alpha\beta} = \tfrac{1}{c}(x^\alpha\mathfrak{T}^{\mu\beta} - x^\beta\mathfrak{T}^{\mu\alpha}) + S^{\mu\alpha\beta} \tag{99.6}$$
$$S^{\mu\alpha\beta} = -\tfrac{1}{4}\hbar\,\tilde{\psi}(\boldsymbol{\gamma}^\mu\boldsymbol{\sigma}^{\alpha\beta} + \boldsymbol{\sigma}^{\alpha\beta}\boldsymbol{\gamma}^\mu)\psi \tag{99.7}$$

With the Dirac spin tensor $S^{\mu\alpha\beta}$ now in hand,[64] we are in position to undertake the symmetrization of $\mathfrak{T}^{\mu\alpha}$. The generic equation (83) becomes

$$\begin{aligned} H^{\sigma\mu\nu} &= -\tfrac{1}{8}\hbar c\tilde{\psi}\big\{\boldsymbol{\gamma}^\sigma\boldsymbol{\sigma}^{\mu\nu} + \boldsymbol{\sigma}^{\mu\nu}\boldsymbol{\gamma}^\sigma + \boldsymbol{\gamma}^\mu\boldsymbol{\sigma}^{\nu\sigma} + \boldsymbol{\sigma}^{\nu\sigma}\boldsymbol{\gamma}^\mu - \boldsymbol{\gamma}^\nu\boldsymbol{\sigma}^{\sigma\mu} - \boldsymbol{\sigma}^{\sigma\mu}\boldsymbol{\gamma}^\nu\big\}\psi \\ &= \tfrac{1}{8}i\hbar c\tilde{\psi}\big\{\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\sigma - \boldsymbol{\gamma}^\sigma\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu\big\}\psi \text{ after algebraic simplifications} \\ &= -H^{\mu\sigma\nu} \quad\text{as required} \end{aligned} \tag{99.8}$$

---

[63] $\boldsymbol{G}^\dagger = \boldsymbol{G}$ and $(\boldsymbol{G}\boldsymbol{\gamma}^\mu)^\dagger = (\boldsymbol{G}\boldsymbol{\gamma}^\mu)$: see again the text preceding (55).

[64] For unaccountable reasons, (99.7) seems seldom to make an appearance in the standard literature, but see N. N. Bogoliubov & D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (1959), eq. (7.31). It was from Chapter I of that volume that I learned my classical field theory, and the treatment of the classical Dirac field presented in §§6 & 7 still seems to me outstanding.

Drawing upon the fundamental anticommutation relation (53.1), one can show by quick calculation that

$$\big\{\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\sigma - \boldsymbol{\gamma}^\sigma\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu\big\} = \begin{cases} 2i\boldsymbol{\sigma}^{\mu\nu}\boldsymbol{\gamma}^\sigma + 2g^{\sigma\mu}\boldsymbol{\gamma}^\nu - 2g^{\sigma\nu}\boldsymbol{\gamma}^\mu \\ 2i\,\boldsymbol{\gamma}^\sigma\boldsymbol{\sigma}^{\mu\nu} - 2g^{\sigma\mu}\boldsymbol{\gamma}^\nu + 2g^{\sigma\nu}\boldsymbol{\gamma}^\mu \end{cases}$$

So we have

$$\begin{aligned}\vartheta_{\mu\nu} \equiv \partial_\sigma H^\sigma{}_{\mu\nu} = \quad & \tfrac{1}{4}i\hbar c\tilde{\psi}\big\{i\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\gamma}^\sigma\psi_{,\sigma} + \boldsymbol{\gamma}_\nu\psi_{,\mu} - \boldsymbol{\gamma}_\mu\psi_{,\nu}\big\} \\ & + \tfrac{1}{4}i\hbar c\big\{i\tilde{\psi}_{,\sigma}\boldsymbol{\gamma}^\sigma\boldsymbol{\sigma}_{\mu\nu} - \tilde{\psi}_{,\mu}\boldsymbol{\gamma}_\nu + \tilde{\psi}_{,\nu}\boldsymbol{\gamma}_\mu\big\}\psi\end{aligned}$$

The field equations $\boldsymbol{\gamma}^\sigma\psi_{,\sigma} = -i\varkappa\psi$ and $\tilde{\psi}_{,\sigma}\boldsymbol{\gamma}^\sigma = +i\varkappa\tilde{\psi}$ are invoked to bring about cancellation of the $\boldsymbol{\sigma}_{\mu\nu}$-factors, leaving

$$\vartheta_{\mu\nu} = -\tfrac{1}{4}i\hbar c\big\{(\tilde{\psi}\boldsymbol{\gamma}_\mu\psi_{,\nu} - \tilde{\psi}_{,\nu}\boldsymbol{\gamma}_\mu\psi) - (\tilde{\psi}\boldsymbol{\gamma}_\nu\psi_{,\mu} - \tilde{\psi}_{,\mu}\boldsymbol{\gamma}_\nu\psi)\big\} \tag{99.9}$$

which is manifestly real and (in this instance) manifestly $\mu\nu$-antisymmetric:

$$\vartheta_{\mu\nu} = \vartheta^*_{\mu\nu} \quad\text{and}\quad \vartheta_{\mu\nu} = -\vartheta_{\nu\mu}$$

We now resolve the stress-energy tensor $\mathfrak{T}_{\mu\nu}$, as described at (99.3), into its symmetric and antisymmetric parts

$$\mathfrak{T}_{\mu\nu} = \tfrac{1}{2}\big(\mathfrak{T}_{\mu\nu} + \mathfrak{T}_{\nu\mu}\big) + \tfrac{1}{2}\big(\mathfrak{T}_{\mu\nu} - \mathfrak{T}_{\nu\mu}\big)$$

and notice that $\frac{1}{2}\big(\mathfrak{T}_{\mu\nu} - \mathfrak{T}_{\nu\mu}\big)$ is precisely the negative of $\vartheta_{\mu\nu}$! Therefore Belinfante's

$$\begin{aligned} T_{\mu\nu} &= \mathfrak{T}_{\mu\nu} + \vartheta_{\mu\nu} \\ &= \tfrac{1}{2}\big(\mathfrak{T}_{\mu\nu} + \mathfrak{T}_{\nu\mu}\big) \quad \text{is assuredly symmetric, and given by} \\ &= \tfrac{1}{4}i\hbar c\Big\{(\tilde{\psi}\boldsymbol{\gamma}_\mu\psi_{,\nu} - \tilde{\psi}_{,\nu}\boldsymbol{\gamma}_\mu\psi) + (\tilde{\psi}\boldsymbol{\gamma}_\nu\psi_{,\mu} - \tilde{\psi}_{,\mu}\boldsymbol{\gamma}_\nu\psi)\Big\} \end{aligned} \tag{99.10}$$

In short: Belinfanate's procedure works... but one must *work* to make it work!

The energy density of a Dirac field, by this accounting, becomes

$$\mathcal{E} = \tfrac{1}{2}i\hbar c\,(\tilde{\psi}\boldsymbol{\gamma}_0\psi_{,0} - \tilde{\psi}_{,0}\boldsymbol{\gamma}_0\psi) \tag{99.12}$$

This is precisely the formula (57) we obtained *prior* to symmetrization:

$$T_{00} = \mathfrak{T}_{00}$$

as implied by the atypical antisymmetry of the $\vartheta_{\mu\nu}$ to which the Dirac theory has just lead us. The occurance of mixed signs in (99.12) casts into doubt any hope that the Dirac theory might conform to the "principle of energy non-negativity," and indeed, it stands in celebrated violation of that principle. This deep fact follows most readily from (99.12), according to which

$$\text{time-reversal sends } \mathcal{E} \longrightarrow -\mathcal{E}$$

and can be attributed to the circumstance that if $(E/c)^2 - p^2 = (mc)^2$ then

$$E = \pm c\sqrt{p^2 + (mc)^2}$$

It was to circumvent this seeming defect of the theory that Dirac (who imagined himself to be inventing a theory of electrons) invented the "electron sea," and invoked the exclusion principle to fill up all the negative energy states. Proper resolution of the problem had to await development of a theory of *quantized* Dirac fields; in that expanded setting it becomes possible to "reinterpret the problem away" (i.e., to assign the "negative energy states" to "antiparticles").

We look finally to the spin density of the classical Dirac field; i.e., to the expressions

$$\begin{aligned} R_1 &\equiv S^{001} \qquad & S_1 &\equiv S^{023}\\ R_2 &\equiv S^{002} & S_2 &\equiv S^{031}\\ R_3 &\equiv S^{003} & S_3 &\equiv S^{012} \end{aligned}$$

Look first to the former: (99.7) supplies $R_1 = \frac{1}{4}\hbar\,\tilde{\psi}(\boldsymbol{\gamma}^0\boldsymbol{\sigma}^{01} + \boldsymbol{\sigma}^{01}\boldsymbol{\gamma}^0)\,\psi$, but

$$\boldsymbol{\gamma}^0\boldsymbol{\sigma}^{01} + \boldsymbol{\sigma}^{01}\boldsymbol{\gamma}^0 = \boldsymbol{\gamma}^0\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1 - \boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^0 + \boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^0 - \boldsymbol{\gamma}^1\boldsymbol{\gamma}^0\boldsymbol{\gamma}^0 = \mathbf{0}$$

Evidently

$$R_1 = R_2 = R_3 = 0$$

This is a remarkable fact, a fact upon which all inertial observers agree... but which they do not much talk about in the literature; it can be phrased this way: *the spin-analogs of* $\{K_1, K_2, K_3\}$*—which in discussion subsequent to (77) were found to refer to the center of mass motion of a field system—are for Dirac fields trivial*. Drawing upon (99.7), we have

$$\begin{aligned} S_1 &= \psi^\dagger\boldsymbol{\Sigma}_1\psi \quad \text{with} \quad \boldsymbol{\Sigma}_1 \equiv -\tfrac{1}{4}\hbar\boldsymbol{G}(\boldsymbol{\gamma}^0\boldsymbol{\sigma}^{23} + \boldsymbol{\sigma}^{23}\boldsymbol{\gamma}^0)\\ S_2 &= \psi^\dagger\boldsymbol{\Sigma}_2\psi \quad \text{with} \quad \boldsymbol{\Sigma}_2 \equiv -\tfrac{1}{4}\hbar\boldsymbol{G}(\boldsymbol{\gamma}^0\boldsymbol{\sigma}^{31} + \boldsymbol{\sigma}^{31}\boldsymbol{\gamma}^0)\\ S_3 &= \psi^\dagger\boldsymbol{\Sigma}_3\psi \quad \text{with} \quad \boldsymbol{\Sigma}_3 \equiv -\tfrac{1}{4}\hbar\boldsymbol{G}(\boldsymbol{\gamma}^0\boldsymbol{\sigma}^{12} + \boldsymbol{\sigma}^{12}\boldsymbol{\gamma}^0) \end{aligned}$$

Recalling from (54) how the $\boldsymbol{\gamma}^\mu$ were defined, and from (55) how $\boldsymbol{G}$ was defined; recalling also the definitions $\boldsymbol{\sigma}^{\mu\mu}$ which were motivated by (99.4), and relying upon *Mathematica* to perform the matrix algebra, we obtain

$$\boldsymbol{\Sigma}_1 = \tfrac{1}{2}\hbar\begin{pmatrix}\boldsymbol{\sigma}_1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}_1\end{pmatrix}, \quad \boldsymbol{\Sigma}_2 = \tfrac{1}{2}\hbar\begin{pmatrix}\boldsymbol{\sigma}_2 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}_2\end{pmatrix}, \quad \boldsymbol{\Sigma}_3 = \tfrac{1}{2}\hbar\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}_3\end{pmatrix}$$

The simplicity of this result is striking, and so are its implications, for it is immediately evident that $\boldsymbol{\Sigma}$-algebra duplicates the Pauli algebra; specifically

$$\begin{aligned}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2 - \boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_1 &= i\hbar\,\boldsymbol{\Sigma}_3\\ \boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_3 - \boldsymbol{\Sigma}_3\boldsymbol{\Sigma}_2 &= i\hbar\,\boldsymbol{\Sigma}_1\\ \boldsymbol{\Sigma}_3\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_3 &= i\hbar\,\boldsymbol{\Sigma}_2\end{aligned}$$

which inform us we are dealing with "angular momentum algebra." Moreover

$$\boldsymbol{\Sigma}_1^2 + \boldsymbol{\Sigma}_2^2 + \boldsymbol{\Sigma}_3^2 = \ell(\ell+1)\boldsymbol{I} \quad \text{with} \quad \ell = \tfrac{1}{2}$$

We have touched here upon ideas standard to elementary quantum mechanics,[65] and it is true, moreover, that the expressions

$$\iiint \psi^\dagger \boldsymbol{\Sigma}_1 \psi\, dx^1 dx^2 dx^3, \text{ etc.}$$

which describe the *total* spin of the Dirac field do resemble quantum mechanical expectation values. It's all the more important, therefore, to appreciate that we are at the moment doing classical field theory—*not* quantum mechanics—and that the $\hbar$'s which appear in our equations are simply abbreviations for a certain constellation

$$\hbar \equiv mc/\varkappa$$

of the dimensioned constants m, c and $\varkappa$ which enter as essential players into the classical theory of Dirac fields; $\hbar$, for present purposes, has *not* the status of a God-given constant of Nature, but might have any value.

Concluding remarks. All linear relativistic field theories assume a distinctly Dirac-like appearance (36) when cast into canonical form. One expects the discussion just concluded to provide, therefore, a pattern adaptable to the generality of such theories. Were one to undertake such a program, one would want in particular to identify the algebraic circumstances which the cases $\mathcal{E} \geqslant 0$ from the cases $\mathcal{E} \gtrless 0$.

In an expanded review of relativistic field theories one would expect to encounter some discussion of "zero-mass theories." We have seen that the limit $\varkappa \downarrow 0$ must be approached circumspectly, that massless fields display internal degrees of freedom (think of gauge freedom in the Maxwellian case) which are absent if $\varkappa \neq 0$. The foundations of the theory of massless fields were laid down

[65] See, for example, D. Griffiths, *Introduction to Quantum Mechanics* (1995), §§4.3 & 4.4.

by Hermann Weyl[66] and, in an alternative form, by Ettore Majorana.[67] Brief accounts of the subject can be found in several of the standard sources.[68]

It is entirely possible to retain Lorentz covariance but *abandon the linearity assumption* which has dominated the preceding discussion. The resulting theory is much more difficult, but for a period during the 1950's Heisenberg imagined it to be the shape of the future.[69] Heisenberg looked to theories of (roughly) the form $\boldsymbol{\gamma}^\mu \partial_\mu \psi + i\varkappa(\tilde{\psi}\boldsymbol{\gamma}_\mu \psi)\boldsymbol{\gamma}^\mu \psi = 0$. Fairly typical of the large (but largely inconsequential) literature is a paper[70] by G. Rosen, who has, for his own reasons, discussed the system $\mathcal{L} = \frac{1}{2} g^{\alpha\beta} \varphi_{,\alpha}\varphi_{,\beta} + g\varphi^6$. Einstein's theory of gravitation provides the non-linear field theory *par excellence*, but there one abandons not only linearity but also special relativity. The theory of solitons presents occasional instances of non-linear relativistic fields, though that theory is for the most part studied non-relativistically, on spaces of reduced dimension.

---

[66] "Elektron und Gravitation," Zeits. für Phys. **56**, 330 (1929). It was entirely characteristic of Weyl to draw his motivation from general relativity, rather than from the still embryonic physics of elementary particles; recall that Pauli's "neutrino hypothesis" (1933) lay still four years in the future. Pauli criticized Weyl's "two component theory" on grounds that it was not parity symmetric, and it was for that precise reason that the theory—in the guise now of a theory of neutrinos—was revived by T. D. Lee & C. N. Yang: "Question of parity conservation in weak interactions," Phys. Rev. **104**, 254 (1956). It was, by the way, in that same classic paper that Weyl lay the foundation for what was to become gauge field theory; see Chapter 5 in L. O'Raifeartaigh, *The Dawning of Gauge Theory* (1997).

[67] "Teoria relativistica di particelle con momento intrinseco arbitrario," Nuovo Cimento **9**, 335 (1932); "Teoria simmetrica dell' elettrone e delpositrone," Nuovo Cimento **14**, 171 (1937). Majorana (1906–1938?) was a brilliant but habitually morose member of the stellar group which condensed around Fermi at the University of Rome in the late 1920's and early 1930's. The mystery of his disappearance, at age 32, has never been solved. Majorana's first paper is "pre-neutrino," but by the time of his second paper he had reason to be well-acquainted with that subject: Fermi had given the neutrino its name, and made it the central player in his "Versuch einer Theorie der β-Strahlen," Zeits. für Phys. **88**, 161 (1934.)

[68] See Schweber,[7] Chapter 5; Corson,[46] §29. The subject is treated also in Section III of W. Pauli & M. Fierz, "On the relativistic wave equations for particles of arbitrary spin in an electromagnetic field," Proc. Roy. Soc. **173A**, 211 (1939) and in G. Uhlenbeck & O. Laport, "Application of spinor analysis to the Maxwell & Dirac equations," Phys. Rev. **37**, 1380 (1931), both of which are classic.

[69] See his *Introduction to the Unified Field Theory of Elementary Particles* (1966). The theory advocated by Heisenberg was for a while embraced—but then publicly denounced—by Pauli.

[70] "Equations of motion in classical non-linear field theories," J. Math. Phys. **8**, 573 (1967).

Note should be made also of the "non-linear electrodynamics" which Max Born and Leopold Infeld were motivated to devevlop in the early 1930's.[71]

Relativistic linear field theory has been cultivated not as mathematical recreation, but because the subject—even prior to quantization—has important physical work to do. One has interest, therefore, not only in the formal design of such theories (the topic which has concerned us) but also in the *solutions* of the resulting field equations. The elaborate technology developed in response to the latter interest was first sketched by Jordan & Pauli,[72] and brought to its modern form by Schwinger.[73] The central objects are certain "invariant functions"—by nature Green's functions, made available by the linearity assumption.[74] The invariant functions developed in association with the classical theory see service also in the quantum theory of fields.

Information relating to each of the topics mentioned above can, as I have indicated, be found in the literature. I conclude with mention of a topic for which I am, on the other hand, able to cite no reference. Belinfante's procedure, as we have seen, acheives $\mathcal{T}^{\mu\nu}$ symmetrization by appeal to the spin structure of the field system. I interpret this to mean that it was the presence of "spin structure" which *served initially to destroy stress-energy symmetry*. Can one, in physical language, say anything illuminating about the *mechanism* by which spin disrupts symmetry? I pose that question as unfinished business, and think that a sharp answer (in—say—the Fermi/Weisskopf tradition) would serve very usefully to deepen our intuitive understanding of the dynamics of fields.[75]

---

[71] Their "Foundations of a new field theory" (Proc. Roy. Soc. **144A**, 425 (1934)) provides good exercise in the methods of classical field theory. Subsequent papers (Proc. Roy. Soc. **147A**, 522 (1934) and **150A**, 141 (1935)) discuss quantization of the theory; they lead to operators descriptive of the center of mass of the "new field" which fail to commute (as position operators are supposed to do). M. H. L. Pryce, in a companion paper ("Commuting coordinates in the new field theory," Proc. Roy. Soc. **150A**, 166 (1934)) shows that commutivity can be restored if one adopts modified definitions which entail addition of a term which refers to the spin of the field. Pryce's construction is in some ways anticipatory of Belinfante's. The Born-Leopold theory is discussed in its historical context by A. Sommerfeld in §37 of his *Electrodynamics* (1952).

[72] P. Jordan & W. Pauli,"Zur Quantenelektrodynamik ladungsfreier Felder," Zeits. für Phys. **47**, 151 (1928).

[73] J. Schwinger, "Quantum electrodynamics. II. Vacuum polarization & self-energy," Phys. Rev. **75**, 651 (1949). The material to which I refer is found in the Appendix.

[74] For surveys of the construction and properties of those functions, see RELATIVISTIC CLASSICAL FIELDS (1973), pp. 156–216; ANALYTICAL METHODS OF PHYSICS (1981), pp. 366–433 or any good quantum fields text; I particularly recommend Appendix A1 in J. M. Jauch & F. Rohrlich, *The Theory of Photons and Electrons* (1955).

[75] H. C. Ohanian's "What is spin?" (AJP **54**, 500 (1986)) takes a first step in the right direction, and is of independent interest.